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Chapter

THE BEHAVIOURS AND PROPERTIES OF MICROSCOPIC PARTICLES IN NONLINEAR SYSTEMS

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ABSTRACT

We here propose the elementary principles of nonlinear quantum mechanics (NLQM), which is based on some problems in quantum mechanics. The motion laws and some main properties of microscopic particles in nonlinear quantum systems are studied in detail using these elementary principles. Concretely speaking, we investigate in this paper the wave-particle duality of the solution of the nonlinear Schrodinger equation, the stability of microscopic particles described by NLQM, invariances and conservation laws of motion of particles, the Hamiltonian principle of particle motion and corresponding Lagrangian and Hamilton equations, the classical rule of microscopic particle motion, the mechanism and rules of particle collision, the features of reflection and the transmission of particles at interfaces, and the uncertainty relation of particle's momentum and position as well as the eigenvalue of particles and its properties, and so on. We obtained the invariance and conservation laws of mass, energy and momentum and angular momentum for the microscopic particles, which are also some elementary and universal laws of matter in the NLQM and give the methods and ways of solving the eigenvules. We also find that the laws of motion of microscopic particles are completely different from that in the linear quantum mechanics (LQM). They have a lot of new properties; for example, the particles possess the real wave-corpuscle duality, obey the classical rule of motion and conservation laws of energy, momentum and mass, satisfy minimum uncertainty relation, can be localized due to the nonlinear interaction, and its position and momentum can also be determined, etc. From these studies, we see clearly that rules and features of microscopic particle motion in NLQM is different from that in LQM, the latter is a especial case of the former at the nonlinear interaction to equal to zero.The NLQM is a new physical theory, and a necessary result of the development of quantum

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mechanics and has a correct representation of describing microscopic particles in nonlinear systems, which can solve problems disputed for about a century by scientists in the LQM field. Hence, the NLQM built is very necessary and correct, can promote the development of physics and can enhance and raise the knowledge and recognition levels to the essences of microscopic matter. We can predict that nonlinear quantum mechanics has extensive applications in physics, chemistry, biology and polymers, etc

Keywords *quantum mechanics, microscopic particle, nonlinear systems, nonlinear Schrodinger equation, basic principle, nonlinear theory, wave-particle duality, motion rule*

1. INTRODUCTION, PHYSICAL BACKGROUND

As is known, the quantum mechanics established by several great scientists such as Bohr, Born, Schrodinger and Heisenberg, etc., in the early 1900s [1-6] is the science describing the properties and rules of motion of microscopic particles (MIP). It is a foundation of modern science, in which the state of microscopic particles is described by the Schrodinger equation:

$$i\mathrm{h}\frac{\partial y}{\partial t} = -\frac{\mathrm{h}^2}{2m}\nabla^2 y + V(\vec{r},t)y \qquad (1)$$

where $\mathrm{h}^2\nabla^2/2m$ is the kinetic energy operator, $V(\vec{r},t)$ is the externally applied potential operator, m is the mass of particles, $y(\vec{r},t)$ is a wave function describing the states of particles, $\vec{r}$ is the coordinate or position of the particle, and t is the time. This description indicates that MIPs have the wave-particle duality because it is both a wave and has a determinant mass. However, equation (1) is a wave equation, and if only the externally applied potential is known, we can find the solutions of the equation. But, for all externally applied potentials, the solutions of the equation are always a linear or dispersive wave, for example, at $V(\vec{r},t)$=0, its solution is a plane wave as follows:

$$y(\vec{r},t) = A'\exp[i(\vec{k}\cdot\vec{r} - wt)] \qquad (2)$$

where k is the wavevector of the wave, w is its frequency, and A' is its amplitude. When $V(\vec{r},t) \neq 0$, its solutions are a de Broglie wave or a Bloch wave, and so on. This means that $y(\vec{r},t)$ denotes only a wave. Therefore, the MIP is represented by a wave in quantum mechanics. It always disperses in total space and cannot be localized. In other words, the solutions of Eq. (1) in the case of any potential possessing only a wave feature and not a particulate nature or corpuscle-wave duality, thus the MIPs is unstable, and have not a determinant position in the space at any time. This is not consistent with the above description of MIP. Thus we have to introduce Born's hypothesis and can to use the

$\left|y(\vec{r},t)\right|^2$ representing the probability occurred particle at position $\vec{r}$. Thus we refer ever to $y(\vec{r},t)$ as a probability wave. As is known, the wave feature and probability concept of MIP is incompatible with the traditional concept of stability and the determinant size of particles [7-9]. The above properties of MIPs result in the occurrence of the probability concept, uncertainty relation, and statistical average values of mechanical quantities in quantum mechanics, which were thought to be an elementary concepts and principle of quantum mechanics and the intrinsic features of MIP, but are all contradictory with regard to particles. Thus, we have reasons to improve and develop quantum mechanics [7-9].

However, why does quantum mechanics have these questions? This is worth studying deeply and in detail. As is known, equation (1) describes the motion of a particle; the corresponding Hamiltonian operator of the system is

$$\hat{H}(t) = \hbar^2\nabla^2/2m + V(\vec{r},t) \qquad (3)$$

Obviously, it consists only of kinetic and potential operator of particles; the potential is only determined by an externally applied field, and not related to the state or wavefunction of the particle, thus the potential can only change the states of MIP, and cannot change its nature and essence. Therefore, the natures and features of MIP are only determined by the kinetic term. Thus there is no force or energy to obstruct and suppress the dispersing effect of kinetic energy in the system, then the MIP disperses and propagates in total space, and cannot be localized at all. This is the main reason why MIP has only wave feature in quantum mechanics. Meanwhile, the Hamiltonian in Eq.(3) does not represent practical essences and features of MIP. In real physics, the energy operator of the systems and number operator of particles are always associated with the states of particles, i.e., they are related to the wavefunction of MIP. On the other hand, Eq.(2) or (3) can describe only the states and feature of a single particle, and cannot describe the states of many particles. However, a system composed of one particle does not exist in nature. The simplest system in nature is the hydrogen atom, but it consists of two particles. In such a case, when we study the states of particles in realistic systems composed of many particles and many bodies using quantum mechanics, we have to use a simplified and uniform average-potential unassociated with the states of particles to replace the complicated and nonlinear interaction among these particles [10-11]. This means that the motions of the particles or background field are completely freezed in such a case. Thus, these complicated effects and nonlinear interactions determining essences and natures of particles are ignored completely. Therefore, the state and properties of particles determined by the simplified or average potential is not real and correct. Obviously,this is not reasonable. Then we can only say that quantum mechanics is an approximate and linear theory and cannot represent completely the properties of motion of MIPs.We here refer to it as linear quantum mechanics (LQM). Meanwhile, a lot of hypotheses or theorems of particles in quantum mechanics also do not agree with conventional understanding, and have excited a long-time debate between scientists. Up to now, there is no unified conclusion. Therefore, it is necessary to improve and develop LQM. However, what is its direction of development? From the above studies we know that a key shortcoming or defect of LQM is its ignoring of dynamic states of other particles or

background field, and the dependence of the Hamiltonian or energy operator of the systems on the states of particles and nonlinear interactions among these particles. As a matter of fact, the nonlinear interactions always exist in any realistic physics systems including the hydrogen atom, if only the real motions of the particles and background as well as their interactions are completely considered[10-15]. At the same time, it is also a reasonable assumption that the Hamiltonian or energy operator of the systems depend on the states of particles[12-25]. Hence, to establish a correct new quantum theory, we must break through the elementary hypotheses of LQM, and use the above reasonable assumptions to include the nonlinear interactions among the particles or between the particles and background field as well as the dependences of the Hamiltonian of the systems on the state of particles.Thus, we must establish nonlinear quantum mechanism (NLQM) to study the rules of motion of MIPs in realistic systems with nonlinear interactions by using the above new idea and method[12-25].

Pang worked out the NLQM describing the properties of motion of MIPs in nonlinear systems [12-27]. The elementary principles, theory, calculated rules and applications of NLQM were described in Pang *et al.*'s books [26, 27]. For the development of quantum mechanics from linear range to nonlinear domain in the basis of original quantum mechanics, Pang worked at and investigated this problem for about 20 years [12-27]. In this investigation, Pang first sought the roots of these problems existing in the LQM. Subsequently, Pang [26-30] broke through the restrictions of the elementary hypotheses for the independence of the Hamiltonian of the systems on the states of the particles and the linearity of the theory in the LQM, and proposed and established the elementary principles and theory of the NLQM, based on the relations among the nonlinear interaction and soliton motions and macroscopic quantum effect, and incorporating modern theories of superconductors, superfluids and solitons, according to the features of macroscopic quantum effects and soliton theory. A lot of practices and experiences demonstrate that the NLQM is successful[26-30]. This paper is essentially composed of four parts. The first presents the fundamental principles of NLQM. The descriptions of essential features of microscopic particles, including the wave-particle duality of the solution of the nonlinear Schrodinger equation, the stability of microscopic particles described by NLQM, the invariances and conservation laws of motion of particles, the Hamiltonian principle of particle motion and corresponding Lagrangian and Hamilton equations, the classical rule of microscopic particle motion, the mechanism and rules of particle collision, the features of reflection and the transmission of particles at interfaces, and the uncertainty relation of particle's momentum and position, are covered in the second part. The eigenvules and eigenequation of the Hamiltonian operator of the systems and nonlinear Schrodinger equations as well as their properties are presented in the thirdpart, A conclusion of the investigation is finally given. These investigations are helpful for understanding the properties of MIPs in nonlinear systems and the essences of NLQM.

2. FUNDAMENTAL PRINCIPLES OF NONLINEAR QUANTUM MECHANICS

Based on the earlier discussion on linear quantum mechanics, the fundamental principles of nonlinear quantum mechanics (NLQM) may be summarized as follows [12-30].

(1) Microscopic particles in a nonlinear quantum system are described by the following wave function,

$$f(\rho,t)=j(\rho,t)e^{iq(\rho,t)} \tag{4}$$

where both the amplitude $j(\rho,t)$ and phase $q(\rho,t)$ of the wave function are functions of space and time, and satisfy different equation of motion.

(2)In the nonrelativistic case, the wave function $f(\rho,t)$ satisfies the generalized nonlinear Schrodinger equation (NLSE), i.e.,

$$i\hbar\frac{\partial f}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2 f\pm b|f|^2 f+V(r,t)f+A(f), \tag{5}$$

or

$$m\frac{\partial f}{\partial t}=-\frac{\eta^2}{2m}\nabla^2 f\pm b|f|^2 f+V(\rho,t)f+A(f) \tag{6}$$

where μ is a complex number, V is an external potential field, A is a function of $f(\rho,t)$, and b is a coefficient indicating the strength of nonlinear interaction.

In the relativistic case, the wave function $f(\rho,t)$ satisfies the nonlinear Klein-Gordon equation (NLKGE), including the generalized Sine-Gordon equation (SGE) and the f^4-field equation, i.e.,

$$\frac{\partial^2 f}{\partial t^2}-\frac{\partial^2 f}{\partial x_j^2}=b\sin f+g\frac{\partial f}{\partial t}+A(f) \text{ (j=1,2,3)} \tag{7}$$

$$\frac{\partial^2 f}{\partial t^2}-\frac{\partial^2 f}{\partial x_j^2}\mu af\pm b|f|^2 f=A(f) \text{ (j=1,2,3)} \tag{8}$$

where γ represents a dissipative or frictional effects, a is a constant, β is a coefficient indicating the strength of nonlinear interaction and A is a function of $f(\rho,t)$.

From the above fundamental principles, we see clearly that the NLQM breaks through the fundamental hypotheses of the LQM in two aspects, namely the linearity of dynamic equations and independence of the Hamiltonian operator with the wave function of the particles. In the NLQM, the dynamic equations are all some nonlinear partial differential equations, in which nonlinear interactions, $b|f|^2 f$, related to state wave function f are involved. Thus we can expect that the Hamiltonian or Lagrangian operators corresponding to these equations also are all related to the state wave function f, which can see in Eqs. (16)-(17). Hence, so far as this point is concerned, the NLQM is really a break-through or a new development in quantum mechanics.

3 THE FEATURES OF MICROSCOPIC PARTICLES IN NONLINEAR QUANTUM MECHANICS

3.1. The Solutions Of Nonlinear Schrodinger Equation And Its Wave-Particle Features

In the one-dimensional case, the solution of Eq.(5) at V(x,t)= A(f)=0 can be found by using some methods, for example, the inverse scattering method, which is of the form[26-27,31]:

$$f(x,t)=A_0\sec h\left\{\frac{A_0\sqrt{b}}{\sqrt{2}\mathrm{h}}\left[\sqrt{2m}(x-x_0)-v_e t\right]\right\}e^{iv_e\left[\sqrt{2m}(x-x_0)-v_c t\right]/2\mathrm{h}} \tag{9}$$

where $A_0=\sqrt{\frac{v_e^2-2v_c v_e}{2b}}$. This solution is completely different from Eq.(2). In fact, it is a bell-type non-topological soliton as shown in Figure1. Therefore, the microscopic particle in NLQM is a soliton. Here, $j(x,t)=A_0\sec h\{A_0[\sqrt{2mb}(x-x_0)-v_e t]/\sqrt{2}\eta\}$ is the envelop of the solution, and $\exp\left\{iv_e\left[\sqrt{2m}(x-x_0)-v_e t\right]/2\eta\right\}$ is its carrier wave. The form of soliton of MIP is shown in Figure1. The envelop φ(x,t) is a slow varying function and the mass center of the particle, the position of the mass center is just at x_0, A_0 is its amplitude, and its width is given by $W=2p\eta/(\sqrt{mb}A_0$. Thus, the size of the soliton is $A_0W=2p\eta/\sqrt{mb}$, and a constant. This shows that the particle has determinant form and size, and is localized, v_e is the group velocity of the particle (soliton), v_c is the phase speed of the carrier wave. For a certain system, v_e and v_c and the size of the particle are determinant and do not change with time. According to the soliton theory [32-33], the bell-type soliton in Eq. (9) can move freely over macroscopic distances in a uniform velocity v_e in space-time retaining its form, energy, momentum and other quasi-particle properties. In this condition, its mass, momentum and energy are some constants, and can be represented by [26-27]

$$N_s=\int_{-\infty}^{\infty}|f|^2dx'=2\sqrt{2}A_0$$

$$p=-i\int_{-\infty}^{\infty}\left(f^*f_{x'}-ff_{x'}^*\right)dx'=2\sqrt{2}A_0v_e=N_sv_e=const$$

$$E=\int_{-\infty}^{\infty}\left[|f_{x'}|^2-\frac{1}{2}|f|^4\right]dx'=E_0+\frac{1}{2}M_{sol}v_e^2 \tag{10}$$

where $x'=x/\sqrt{\eta^2/2m}$, $t'=t/\eta$, and $M_{sol}=N_s=2\sqrt{2}A_0$ is effective mass of MIP, which is a constant. The energy, mass and momentum of the particle are invariant and cannot be dispersed in its motion. Just so, the position vector $\vec{r}$ and position x (or x,y,z) has definitively physical significance, and denotes exactly the positions of MIPs at time t. Thus, the wavefunction $f(\vec{r},t)$ or φ(x,t) can represent exactly the states of MIP at the position $\vec{r}$ or x and time t. This is consistent with the concept of particles or corpuscles. In such a case the effective potential with two minimum possessed by the particle is shown in Fig.1c. At the same time, in Figure 1(d), we show the collision property of two soliton solutions of Eq. (9) by numerical simulation technique. From this figure, we see clearly that the two particles can go through each other while retaining their form after the collision, which is the same with that of the classical particles. Therefore, the microscopic particle in NLQM has an obvious particulate feature. However, the envelope of the solution in Eq. (9) is a solitary wave. It has a certain wavevector and frequency as shown in Figure 1(b), and can propagate in space accompanying the carrier wave, i.e., the carrier wave carries the envelope to propagate in space-time; the feature of propagation depends only on the concrete nature of MIPs. Figure 1(b) shows the width of the frequency spectrum of the envelope $\varphi(x,t)$, the frequency spectrum has a localized structure around the carrier frequency ω_0. If $V(x,t)\neq 0$, we can find also out similar soliton solutions with Eq.(9), where the differences are only the amplitudes and velocities. So, the microscopic particle in NLQM has exactly wave-particulate duality [26-27]. This consists of Davisson and Germer's experimental result of electron diffraction on double seam in 1927.

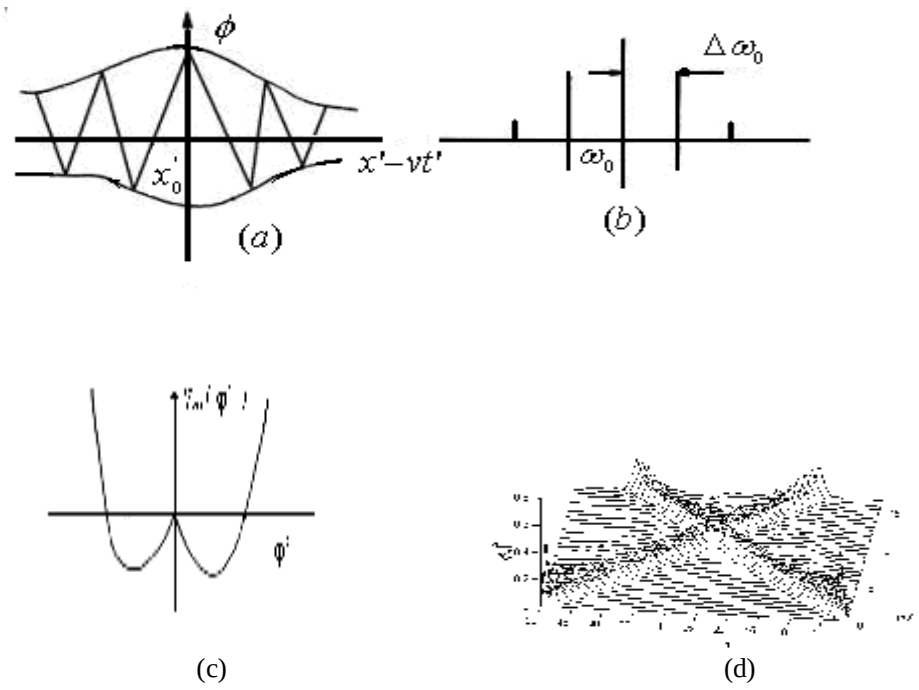

Figure 1. The solution in Eq. (9) at V=A=0 in Eq. (5) and its features.

However, we must remember that the solution of dynamic equations (5)-(8) in nonlinear quantum mechanics in the limits of weak nonlinear interaction also are not exactly the solutions of the dynamic equations in linear quantum mechanics. To see this clearly, we first examine the velocity of the skirt of the soliton given in Eq. (9), which is now rewritten as

$$f(x',t') = 2\sqrt{2}k \operatorname{sech}\left[2k\left(x' - x_0' - u_e t'\right)\right] e^{iu_e\left(x' - x_0' - u_c t'\right)/2} \tag{11}$$

for $b = 1, V(\vec{r},t') = A(f) = 0$ in Eq. (5). As is known, the nonlinear term in Eq. (5) sharpens the peak, while its dispersion term has the tendency to leave it off. Then, for weak nonlinear interaction and small skirt $f(x',t')$, it may be approximated by (for $x > u_e t$)

$$f = 4\sqrt{2}k e^{-2k(x' - u_e t')} e^{iu_e(x' - u_c t')/2} \tag{12}$$

and the small term $|f|^2 f$ in Eq. (5) in such a case can be approximated by

$$if_{t'} + f_{x'x'} \approx 0 \tag{13}$$

Substituting Eq. (12) into Eq. (13), we get $u_e \approx 4k$, which is the group speed of the particle. (Near the top of the peak, we must take both the nonlinear and dispersion terms into account because their contributions are of the same order. The result is the group speed.). Here, we have only checked the formula for the region where $f(x,t)$ is small; that is, when a particle is approximated by Eq.(12), it satisfies the approximate wave equation (13) with $u_e \approx 4k$.

However, if Eq.(13) is treated as a linear Schrodinger equation, its solution is a plane wave as follows:

$$f'(x,t) = Ae^{i(kx - wt)} \tag{14}$$

We now have $w = k^2$, which gives the phase velocity w/k as $u_c = k$ and the group speed $\partial w/\partial k = u_{\mathrm{gr}} = 2k$. Apparently, this is different from $u_e = 4k$. This is because the solution Eq. (12) is essentially from Eq. (14). This shows clearly that the solution Eq. (14) of linear Schrodinger equation also is not the solution of nonlinear Schrodinger equation (5) with V(x,t)= A(f)=0 in the case of weak nonlinear interactions. Hence, nonlinear quantum mechanics differs in essence from linear quantum mechanics. Solution Eq. (12) is a "divergent solution" (i.e., $f(x,t) \to \infty$ at $x \to -\infty$), which is not an "ordinary plane wave". The concept of group speed does not apply to a divergent wave. Thus, we can say that the soliton is made from a divergent solution, which is abandoned in the linear waves. The divergence develops by the nonlinear term to yield solitary waves of finite amplitude. When the nonlinear term is very weak, the soliton will diverge, but cannot absolutely become a

plane wave; and if we suppress divergence no soliton will result. These circumstances are clearly seen from the following soliton in the case of nonlinear coefficient $b \neq 1$

$$f(x,t) = 2\sqrt{\frac{2}{b}}k\,\mathrm{sech}\left[2k\left(x' - u_e t'\right)\right]e^{iu_e\left(x' - u_c t'\right)/2} \tag{15}$$

If the nonlinear term approaches zero ($b \to 0$), the solitary wave diverges $\left(f(x,t) \to \infty\right)$. If we want to suppress the divergence, then we have to set $k = 0$. In such a case, we get Eq. (14) from Eq. (15). This illustrates that the nonlinear Schrödinger equation or nonlinear quantum mechanics can reduce to the linear Schrödinger equation or linear quantum mechanics if and only if the nonlinear interaction and the group speed of the particle are zero. Therefore, we can conclude that the particles (solitons) of nonlinear quantum mechanical equations in the weak nonlinear interaction limit is not the same as that in linear quantum mechanics. If the nonlinear interaction is zero, the nonlinear quantum mechanics reduce to the linear quantum mechanics. However, real physical systems or materials are made up of a great number of MIPs, and nonlinear interactions always and widely exist in the systems, even though in the systems of two bodies, such as hydrogen atom. The nonlinear interactions arise from interactions among the MIPs or between the MIPs and the background fields. The nonlinear quantum mechanics should be the correct and more appropriate theory for real systems. It should be used often and extensively, even in weak nonlinear interaction cases. The linear quantum mechanics, on the other hand, is an approximation to the more general nonlinear quantum theory and can be used to study motions of MIPs in systems in which there exists only very weak and negligible nonlinear interactions [26,27].

3.2. DEMONSTRATION OF STABILITY OF MICROSCOPIC PARTICLE

As is known, in classical physics the macroscopic particles are certainly stable. Stability is an elementary feature of a particle. However, is the microscopic particle (MIP) described by NLQM or the solution of nonlinear Schrodinger equation (NLSE) in Eq. (5), for instance, Eq.(9), stable? This is also a basic problem in NLQM, and need to be proved further. In the absence of an externally applied field, the stability of the MIPs in NLQM can be demonstrated by means of the initial and structural stabilities. However, how are MIP's behaviors exposed in an externally applied field? If the motion of all the MIPs is located in a finite range where the potential is lowest, we can say that the MIPs are stable according to the minimum theorem of energy. As a matter of fact, when there are a lot of particles in the system, the interactions with one another among the particles are very complicated; it is very difficult to define the behavior of each one individually. Therefore, we cannot adopt again the strategies of initial stability and collision to study their stability. Instead, we take advantage of the following consideration: when a mechanical system is in a state of minimal energy we may say that it is stable, and to change this state, external energy must be supplied. Pang used this minimal energy consideration to demonstrate the stability of the MIPs as follows.

Let $f(x,t)$ represent the field of the particle; meanwhile, we assume that it possess derivatives of all orders, and all integrations for it be convergent and finite. The Lagrange density function corresponding to the NLSE Eq.(5) at $A(f)=0$ is given as follows:

$$\mathrm{L} = \frac{i\mathrm{h}}{2}\left(f^{*}f_{\mathrm{t}} - ff_{\mathrm{t}}^{*}\right) - \frac{\mathrm{h}^{2}}{2m}\left(\nabla f\cdot\nabla f^{*}\right) - V(x)f^{*}f + b/2\left(f^{*}f\right)^{2} \tag{16}$$

The momentum density of this field is defined as P= $\partial\mathrm{L}/\partial f$. Thus, the Hamiltonian density of the field is as follows

$$\mathrm{H} = \frac{\mathrm{i\,h}}{2}\left(f^{*}\partial_{\mathrm{t}}f - f\partial_{\mathrm{t}}f^{*}\right) - \mathrm{L} = \frac{\mathrm{h}^{2}}{2\,\mathrm{m}}\left(\nabla f\cdot\nabla f^{*}\right) + V(x)ff^{*} - b/2\left(ff^{*}\right)^{2} \tag{17}$$

From Eqs.(16)-(17), we see clearly that the Lagrangian and Hamiltonian operators of the systems corresponding to Eq. (5) are all related to the state wave function of particles and involve all nonlinear interactional energy, b/2 $(ff^{*})^{2}$ related to the states of MIP. This is in essence different from Eq. (3) in LQM. Then the natures and features of MIP are simultaneously determined by the kinetic and nonlinear interaction terms in nonlinear quantum mechanics. Just so, there is a force or energy to obstruct and suppress the dispersing effect of kinetic energy in the system, thus the MIP cannot disperse and propagate in total space, and is localized all the time. This is just the essential reason that the MIP has a particulate nature or corpuscle-wave duality as mentioned above in Section 3.2 in nonlinear quantum mechanics. Therefore, we can say that the above fundamental principles of the NLQM in Eqs.(4)-(8) breaks through the fundamental hypothesis for the independence of Hamiltonian operator with the wave function of the particles in the LQM. This is a new development.

In the general case, the total energy of the particles is a function of t' and is represented by

$$E(t') = \int_{-\infty}^{\infty}\left[\left|\frac{\partial f}{\partial x'}\right|^{2} - \frac{b}{2}\left|ff^{*}\right|^{2} + V(x')\left|f\right|^{2}\right]dx' \tag{18}$$

However, in this case, b and $V(x')$ are not functions of t', where $x'=x/\sqrt{\eta^{2}/2m}$, $t'=t/\eta$. So, the total energy of the systems is a conservative quantity, i.e., $E(t')$=E=const., as shown in Eq.(10). We can demonstrate that when $x'\to\pm\infty$, the solutions of Eq.(5) at $A(f)$=0 and $f(x',t')$ should tend to zero rapidly[26,27], i.e.,

$$\lim_{|x'|\to\infty} f(x',t') = \lim_{|x|\to\infty}\frac{\partial f}{\partial x'} = 0$$

Then

$\int_{-\infty}^{\infty} f^* f dx' = const.$ or a function of t'

The position of mass centre of the particles f can be represented as $\langle x' \rangle = x'_g = x_0 = \int_{-\infty}^{\infty} f'^* f dx' / \int_{-\infty}^{\infty} f^* f dx'$. Thus, the velocity of mass centre of the particles can be denoted by

$$n_g = \frac{d\langle x' \rangle}{dt'} = \frac{d}{dt'}\left\{\frac{\int f * x' f dx'}{\int f * f dx'}\right\} = -2i\frac{\int_{-\infty}^{\infty} f * \frac{\partial f'}{\partial x'} dx'}{\int f * f dx'} \tag{19}$$

However, for different solutions of the same NLSE (5), $\int_{-\infty}^{\infty} f^* f dx'$, $\langle x' \rangle$ and dx'/dt' can have different values. Therefore, it is unreasonable to compare the energy between a definite solution and other solutions. We should compare the energy of one particular solution to that of another solution. The comparison is only meaningful for many MIP systems that have the same values of $\int_{-\infty}^{\infty} f^* f dx' = k.$, $\langle x' \rangle = u$ and $d < x' > / dt' = v$ at the same time t'_0. Based on these, we can determine the stability of the soliton solutions of Eq. (5), for example, Eq. (9). Thus, we assume that the different solutions of the NLSE (5) at $A(f)$=0 satisfy the following boundary conditions at definite time t'_0:

$$\int_{-\infty}^{\infty} f^* f dx' = k\,,\ \langle x' \rangle\big|_{t'=t'_0} = u(t'_0),\ \left.\frac{d < x' >}{dt'}\right|_{t'=t'_0} = v(t'_0) \tag{20}$$

Now we assume the solution of NLSE (5) at $A(f)$=0 to be of the form:

$$f(x',t') = j(x',t')e^{iq(x',t')} \tag{21}$$

Substituting Eq.(21) into Eq.(18), we obtain the energy formula:

$$E = \int_{-\infty}^{\infty}\left[\left(\frac{\partial j}{\partial x'}\right)^2 + j^2\left(\frac{\partial q}{\partial x'}\right)^2 - bj^4 + V(x')j^2\right]dx' \tag{22}$$

Eq. (20) becomes

$$\int_{-\infty}^{\infty} j^2 dx' = k\,,\quad \frac{\int_{-\infty}^{\infty} x' j^2 dx'}{\int_{-\infty}^{\infty} j^2 dx'} = u\left(t_0'\right),\quad \frac{2\int_{-\infty}^{\infty} j^2 \dfrac{\partial q}{\partial x'} dx'}{\int_{-\infty}^{\infty} j^2 dx'} = \dot{u}(t_0') \tag{23}$$

Finding the extreme value of the functional Eq. (22) under the boundary conditions Eq. (23) by means of the Lagrange uncertain factor method, we obtain the following Euler equations:

$$\frac{\partial^2 j}{\partial (x')^2} = \left\{ \begin{matrix} V(x') + C_1(t_0')C_2(t_0')[x' - u(t_0')] + \\ C_3(t_0')\left[2\dfrac{\partial q}{\partial t'} - \dot{u}(t_0')\right] + \left(\dfrac{\partial q}{\partial t'}\right)^2 \end{matrix} \right\} j - b j^3 = 0 \tag{24}$$

$$\frac{\partial^2 j}{\partial (x')^2} j^2 + 2\frac{\partial q}{\partial t'} j \frac{\partial j}{\partial t'} + 2C_3(t_0') j \frac{\partial j}{\partial t'} = 0 \tag{25}$$

where the Lagrange factors C_1 , C_2 and C_3 are all functions of t' . Now, let $C_3(t_0') = -\dfrac{1}{2}\dot{u}(t_0')$

If $2\dfrac{\partial q}{\partial x'} - \dot{u}(t_0') \neq 0$

we can get from Eq.(25)

$$\frac{2}{j}\frac{\partial j}{\partial x'} = \frac{-\dfrac{\partial^2 q}{\partial x'^2}}{-\dfrac{\partial q}{\partial x'} - \dfrac{1}{2}\dot{u}(t_0')}$$

Integration of the above equation yields

$$j^2 = \frac{g(t')}{\dfrac{\partial q}{\partial x'} - \dfrac{1}{2}\dot{u}(t_0')} \text{ or } \left.\frac{\partial q}{\partial x'}\right|_{t'=t_0'} = \frac{g(t_0')}{j^2} + \frac{\dot{u}(t_0')}{2} \tag{26}$$

where $g(t_0')$ is an integral constant. Thus,

$$q(x',t')=g(t_0')\int_0^x \frac{dx'}{j^2}+\frac{\dot{u}(t_0')}{2}x'+M(t_0') \tag{27}$$

Here, $M(t_0')$ is also an integral constant. Again, let

$$C_2(t_0')=\frac{1}{2}\dot{u}(t_0') \tag{28}$$

Substituting Eqs.(26)-(28) into Eq.(24), we obtain

$$\frac{\partial^2 j}{\partial(x')^2}=\left\{V(x')+\frac{\ddot{u}(t_0')}{2}x'+\left[C_1(t_0')-\frac{\dot{u}(t_0')}{2}u(t_0')+\frac{u^2(t_0')}{4}\right]\right\}j-bj^3+\frac{g^2(t_0')}{j^3} \tag{29}$$

Letting

$$C_1(t_0')=\frac{u(t_0')\dot{u}(t_0')}{2}-\frac{\dot{u}^2(t_0')}{2}+M(t_0')+b' \tag{30}$$

where b' is an undetermined constant, which is a function of t'-independent, and assuming $Z=x'-u(t_0')$, then

$$\frac{\partial^2 j}{\partial(x')^2}=\frac{\partial^2 j}{\partial Z^2}$$

is only a function of Z. To make the right-hand side of Eq.(30) be also a function of Z, the coefficients of j, j^3 and $1/j^3$ must also be functions of Z, thus, $g(t_0')=g_0=const.$, and

$$V(x')+\frac{\ddot{u}(t_0')}{2}x'+M(t_0')-\frac{u^2(t)}{4}=\tilde{V}_0(Z)$$

Then, Eq.(29) becomes

$$\frac{\partial^2 j}{\partial(x')^2}=\left\{\tilde{V}[x'-u(t_0')]+b'\right\}j-bj^3+\frac{g^2(t_0')}{j^3} \tag{31}$$

Since $\tilde{V}(Z)=\tilde{V}_0[x'-u(t_0')]=0$ in the present case. Hence, Eq.(31) becomes

$$\frac{\partial^2 j}{\partial(x')^2} = b'j - bj^3 + \frac{g^2(t_0')}{j^3} \tag{32}$$

Therefore, j is the solution of Eq.(32) for the parameters b' =constant and $g(t_0')$=constant. For sufficiently large $|Z|$ we may assume that $|j| \le \tilde{b}/|Z|^{1+\Delta}$, where Δ is a small constant. However, in Eq.(32) we can only retain the solution $j(Z)$ corresponding to $g(t_0')$=0 to ensure that $Lim_{|Z|\to\infty} d^2 j / dZ^2 = 0$, thus, Eq. (32) becomes

$$\frac{\partial^2 j}{\partial(x')^2} = b'j - bj^3 \tag{33}$$

As a matter of fact, if $\partial q/\partial t' = 0$, and considering Eqs.(30)-(31) we can verify that the solution in Eq.(9) can satisfy Eq.(33). In such a case, it is not difficult to show that the energy corresponding to the solution Eq.(9) of Eq.(23) has a minimal value under the boundary conditions of Eq.(23)[26,27]. Thus, we can conclude that the soliton solution of NLSE (5), or the MIP in NLQM is stable in such a case.

3.3. THE CONSERVATION LAWS OF MASS, ENERGY AND MOMENTUM OF PARTICLES

It is known from classical physics that the invariance and conservation laws of mass, energy and momentum and angular momentum are some elementary and universal laws of matter including classical particles in nature. We demonstrate here also that the microscopic particles described by the nonlinear Schrodinger equation in nonlinear quantum mechanics also have such properties. They satisfy the conventional conservation laws of mass, momentum and energy. This shows that the microscopic particles in the nonlinear quantum mechanics also have a corpuscle feature. Therefore, the proposed nonlinear quantum mechanical theory reflects the common rules of motions of matter in nature. To solve this problem, we first should give [26-27] the Lagrangian L= $\int L dx$, where L is denoted by Eq. (16) and Hamiltonian H= $\int H\, dx$, where H is Eq.(17), for the systems corresponding to Eq. (5), respectively. Thus, the number density, the number current, the densities of momentum and energy for the particle can be defined by

$$\left.\begin{aligned} r &= |f|^2,\ p = -i\hbar(f^* f_x - f f_x^*) \\ J &= i\hbar(f^* f_x - f f_x^*),\ \in = \frac{\hbar^2}{2m}|f_x|^2 - \frac{b}{2}|ff^*|^2 + V(x)|f|^2 \end{aligned}\right\} \tag{34}$$

where $f_x = \frac{\partial}{\partial x} f(x,t), f_t = \frac{\partial}{\partial t} f(x,t)$. From Eq. (5) and its conjugate equation as well as Eqs.(16)- (17) and (34) we can obtain

$$\frac{\partial p}{\partial t'} = \frac{\partial}{\partial x'}[2(\frac{\partial f}{\partial x'})^2 + (b|ff^*|^2 - 2V|f|^2 - (f^* \frac{\partial^2}{\partial x'^2} f + f \frac{\partial^2}{\partial x'^2} f^*) + 2iV(f^* \frac{\partial f}{\partial x'})],$$

$$\frac{\partial r}{\partial t'} = \frac{\partial J}{\partial x'}, \frac{\partial \in}{\partial x'} = \frac{\partial}{\partial x'}[r p + i(\frac{\partial f^*}{\partial x'} \frac{\partial^2 f}{\partial x'^2} - \frac{\partial f}{\partial x'} \frac{\partial^2 f^*}{\partial x'^2}) - iV(f^* \frac{\partial f}{\partial x'} - f \frac{\partial f^*}{\partial x'})] \qquad (35)$$

here $x' = x/\sqrt{\eta^2/2m}$, $t' = t/\eta$.Thus, we get the following forms for the integral of motion

$$\frac{\partial}{\partial t'} M = \frac{\partial}{\partial t'} \int r dx' = 0, \frac{\partial}{\partial t'} P = \frac{\partial}{\partial t'} \int p dx' = 0, \frac{\partial E}{\partial t'} = \frac{\partial}{\partial t'} \int \in dx' = 0, \qquad (36)$$

These formulae represent just the conservation of mass, momentum and energy in such a case. This shows that the mass, momentum and energy of the particles (solitons) in the nonlinear quantum mechanical systems still obey general rules of conservation of matter in physics. In the case of V(x,t)=constant, we can find out easily the values of mass, momentum and energy of the particles of Eq.(5) [26-27], as are shown in Eq.(10).

3.4. The Invariance and General Conservation Laws of Particles Described by Nonlinear Schrodinger Equation

We have learned from Eqs.(31)－(34) that some conservation laws for microscopic particles described by the nonlinear *Schrödinger* equation (5) in nonlinear quantum mechanics are always related to the invariance of the action relative to several groups of transformations through the Noether theorem in light of Gelfand and Fomin's (1963) and Bulman and its Kermel's (1989) ideas (see C. Sulem and P. L. Sulem *et al.*'s book and references therein[34,26-27]). Therefore, we first give the Noether theorem for nonlinear *Shrödinger* equation (5) at $A(f) = 0$ according to C.Sulem and P.L.Sulem's method[34].

According to the Lagrangian Eq. (16) of the nonlinear *Shrödinger* equation (5) at $A(f) = 0$, the action of the system can be represented by

$$A\{f\} = \int_{t_0}^{t_1} \int L'(f, \nabla f, f_{t'}, f^*, \nabla f^*, f^*_{t'}) dx' dt' \qquad (37)$$

where L' = L is the Lagrange density function in Eq.(16). For convenience of calculation, we here introduce the following notations:

$\bar{z} = (t', x') = (z_0, z_1, \mathrm{L}\ , z_a)\ \partial_0 = \partial_t, \partial = (\partial_0, \partial_1, \mathrm{L}\ , \partial_d)$ and $\Phi = (\Phi_1, \Phi_2) = (f, f^*)$

where $x' = x/\sqrt{\eta^2/2m}$, $t' = t/\eta$. Thus the action now becomes

$$A\{f\} = \int_D \int_{x'}^{\infty} L'(\Phi, \partial\Phi)\, d\bar{z} \tag{38}$$

Under the action of a transformation T^e which depends on the small parameter e, we have $\bar{z} \to \tilde{z}(z, \Phi, e), \Phi \to \tilde{\Phi}(z, \Phi, e)$, where $\tilde{z}$ and $\tilde{\Phi}$ are assumed to be differentiable with respect to e. When $e = 0$, the transformation reduces to the identity. For infinitesimally small e, we have $\tilde{z} = z + dz, \tilde{\Phi} = \Phi + d\Phi$. At the same time, $T^e, \Phi(z) \to \tilde{\Phi}(\tilde{z})$ by the transformation group T^e, and the domain of integration D is transformed into $\tilde{D}$, then have

$$A\{f\} \to \tilde{A}\{\tilde{f}\} = \int_D \int_{x'}^{\infty} L'(\tilde{\Phi}, \tilde{\partial}\tilde{\Phi})\, d\tilde{z}$$

where $\tilde{\partial}$ denotes differentiation with respect to $\tilde{z}$. Obviously, the change $dA = \tilde{A}\{\tilde{f}\} - A\{f\}$ in the limit of e under the above transformation can be expressed as

$$dA = \int_D \int_{x'}^{\infty} \left[L'(\tilde{\Phi}, \tilde{\partial}\tilde{\Phi}) - L'(\Phi, \partial\Phi) \right] d\bar{z} + \int_D \int_{x'}^{\infty} L'(\Phi, \partial\Phi) \sum_{u=0}^{d} \frac{\partial dz_u}{\partial z_u}\, dz \tag{39}$$

where we used the Jacobian expansion $\dfrac{\partial(\bar{z}_0, \ldots, \bar{z}_d)}{\partial(z_0, \ldots, z_d)} = 1 + \sum_{v=0}^{d} \dfrac{\partial dz_u}{\partial z_u}$, and $L'(\tilde{\Phi}, \tilde{\partial}\tilde{\Phi})$, in the second term on the right-hand side has been replaced by the leading term $L'(\Phi, \partial\Phi)$ in the expansion. Now define

$$\begin{aligned} d\tilde{\Phi}_i &= \tilde{\Phi}_i(\bar{z}) - \tilde{\Phi}_i(\ ) = \partial_u \Phi_i dz_u + d\Phi_i(z) \\ \tilde{\partial}_u \tilde{\Phi}_i(\bar{z}) - \partial_u \Phi_i(z) &= (\tilde{\partial}_u - \partial_u)\tilde{\Phi}_i(\bar{z}) + \partial_u \left[\tilde{\Phi}_i(\bar{z}) - \Phi_i(z) \right] \end{aligned} \tag{40}$$

with

$$\partial_u = \frac{\partial \bar{z}_m}{\partial z_u} \tilde{\partial}_m = \tilde{\partial}_u + \frac{\partial dz_m}{\partial z_u} \tilde{\partial}_m$$

We then have

$$L'\left(\bar{\Phi},\bar{\partial}\bar{\Phi}\right)-L'\left(\Phi,\partial\Phi\right)=\frac{\partial L'}{\partial\Phi_i}\left[\bar{\Phi}_i\left(\bar{z}\right)-\Phi_i\left(z\right)\right]+\frac{\partial L'}{\partial\left(\partial_u\Phi_i\right)}\left[\bar{\partial}_u\bar{\Phi}_i\left(\bar{z}\right)-\partial_m\Phi_i\left(z\right)\right]$$

$$=\frac{\partial L'}{\partial\Phi_i}\partial\Phi_i+\partial_m\left(L'd_u\right)-L'\frac{\partial dz_u}{\partial z_u}+\partial_u\left[\frac{\partial L'}{\partial\left(\partial_n\Phi_i\right)}d\Phi_i\right]-\partial_n\left[\frac{\partial L'}{\partial\left(\partial_u\Phi_i\right)}\right]d\Phi_i$$

Eq. (39) can now be replaced by

$$dA=\int_D\int_{x'}^{\infty}\left\{\frac{\partial L'}{\partial\Phi_i}-\frac{\partial}{\partial z_u}\left[\frac{\partial L'}{\partial\left(\partial_u\Phi_i\right)}\right]\right\}d\Phi_i\,\mathrm{d}z+\int_D\int_{x'}^{\infty}\frac{\partial}{\partial z_u}\left[L'dz_u+\frac{\partial L'}{\partial\left(\partial_u\Phi_i\right)}d\Phi_i\right]\mathrm{d}z \quad (41)$$

where we have used

$$\frac{\partial}{\partial z_u}\left(L'dz_u\right)=L'\frac{\partial dz_u}{\partial z_u}+\frac{\partial L'}{\partial\Phi_i}\partial_u\Phi_i dz_u+\frac{\partial^2 L'}{\partial\left(\partial_n\Phi_i\right)}\partial^2{}_{um}\Phi_i dz_u,$$

$$\frac{\partial L'}{\partial\left(\partial_u\Phi_i\right)}\partial_u d\Phi_i=\frac{\partial}{\partial z_u}\left[\frac{\partial L'}{\partial\left(\partial_u\Phi_i\right)}d\Phi_i\right]-\frac{\partial}{\partial z_u}\left[\frac{\partial L'}{\partial\left(\partial_u\Phi_i\right)}d\Phi_i\right]d\Phi_i$$

Using the Euler-Lagrange equation, the first term on the right-hand side in the equation of dA vanishes. We can get the Noether theorem, *i.e.*, if the action Eq. (38) is invariant under the infinitesimal transformation of the dependent and independent variables $f\to f+df, z\to+dz$ where $z=\left(t,x_1\ldots x_{\mathrm{d}}\right)$, the following conservation law holds[34,28]

$$\frac{\partial}{\partial x_u}\left[L'dx_u-\frac{\partial L'}{\partial\left(\partial_u\Phi_i\right)}d\Phi_i\right]=0,\text{or},\frac{\partial}{\partial x_u}\left[L'dx_u+\frac{\partial L'}{\partial\left(\partial_u\Phi_i\right)}\left(d\Phi_i-\frac{\partial\Phi_i}{\partial z_m}dx_m\right)\right]=0 \quad (42)$$

in terms of $d\hat{\Phi}_i$ defined in Eq.(40).

If the action is invariant under the infinitesimal transformation

$$t'\to\bar{t}=t'+dt'\left(x',t',f\right),x'\to\bar{x}=x'+dx'\left(x',t',f\right),$$

$$f\left(x',t'\right)\to\bar{f}\left(\bar{t},\bar{x}\right)=f\left(t',x'\right)+df\left(t',x'\right),$$

Then, C. Sulem et al[34] obtained that

$$\int\left[\frac{\partial L'}{\partial f_{t'}}\left(\partial_{t'}f\partial_{t'}+\nabla f\cdot d\bar{x}-df\right)+\frac{\partial L'}{\partial f^*_{t'}}\left(\partial_{t'}f^*\partial_{t'}+\nabla f^*\cdot d\bar{x}-df^*\right)-L\delta t'\right]dx' \qquad (43)$$

is a conserved quantity.

For the nonlinear *Schrödinger* equation in Eq.(5) with $A(f)=0$ in the nonlinear quantum mechanics, we have

$$\frac{\partial L'}{\partial f_{t'}}=\frac{i}{2}f^*,\text{and }\frac{\partial L'}{\partial f^*_{t'}}=-\frac{i}{2}f$$

where L' = L is given in Eq.(16). C. Sulem et al[34] obtained the some conservation laws and invariance from the Noether theorem.

a) Invariance under time translation and energy conservation law

The action Eq.(38) is invariant under the infinitesimal time translation $t'\to t'+dt'$ with $dx'=df=df^*=0$, then equation (42) becomes

$$\partial_{t'}\left[\nabla f\cdot\nabla f^*-\frac{b}{2}\left(ff^*\right)^2+V(x',t')f^*f\right]-\nabla\cdot\left(f_{t'}\nabla f^*+f^*_{t'}\nabla f\right)=0$$

This results in the conservation of energy

$$E=\int\left(\nabla f\cdot\nabla f^*-\frac{b}{2}\left(f^*f\right)^2+V(x',t')f^*f\right)dx'=constant$$

b) Invariance of the phase shift or gauge invariance and mass conservation law

It is very clear that the action related to the nonlinear *Schrödinger* equation is invariant under the phase shift $\bar{f}=e^{iq}f$, which for infinitesimal q gives $df=iqf$, with $dt'=dx'=0$. In this case, Eq. (18) becomes

$$\partial_{t'}|f|^2+\nabla\left\{i\left(f\nabla f^*-f^*\nabla f\right)\right\}=0 \qquad (42)$$

This results in the conservation of mass or number of particles.

$$N=\int|f|^2dx'=\text{constant}$$

and the continuum equation

$$\frac{\partial N}{\partial t'} = \nabla \cdot \vec{j},$$

where $\vec{j}$ is the mass current density

$$\vec{j} = -i\left(f\nabla f^* - f^*\nabla f\right)$$

c) Invariance of space translation and momentum conservation law

If the action is invariant under an infinitesimal space translation $x' \to x' + dx'$ with $dt' = df = df^* = 0$, then Eq.(40) becomes

$$\partial_{t'}\left[i\left(f\nabla f^* - f^*\nabla f\right) + \nabla \cdot \left\{2\left(\nabla f^* \times \nabla f + \nabla f \times \nabla f^* + L\right)\right\}\right] = 0$$

This leads to the conservation of momentum

$$\vec{P} = i\int\left(f\nabla f^* - f^*\nabla f\right)dx' = \text{constant}.$$

Note that the center of mass of the microscopic particles is defined by

$$\langle x' \rangle = \frac{1}{N}\int x'|f|^2\, dx',$$

We then have

$$N\frac{d\langle x' \rangle}{dt'} = \int x'\partial_{t'}|f|^2\, dx' = -\int x'\nabla\left[i\left(f\nabla f^* - f^*\nabla f\right)\right]dx'$$
$$= \int i\left(f\nabla f^* - f^*\nabla f\right)dx' = \vec{P} = -\vec{J} = -\int \vec{j}dx' \qquad (44)$$

This is the definition of momentum in classical mechanics. It shows clearly that the microscopic particles described by the nonlinear *Schrödinger* equation have the feature of classical particles.

d) Invariance under space rotation and angular momentum conservation law.

If the action Eq. (38) is invariant under a rotation of angle dq around an axis $\vec{I}$ such that $dt = df = df^* = 0$ and $d\vec{x}' = dq\vec{I} \times \vec{x}'$, this leads to the conservation of the angular momentum

$$\vec{M} = i\int \vec{r}'\times\left(f^*\nabla f - f\nabla f^*\right)dx'$$

Besides the above, C.Sulem et al[34] also derived another invariance of the nonlinear Schrödinger equation from the Noether theorem for nonlinear Schrodinger equation.

e) Galilean Invariance

If the action is invariant under the Galilean transformation

$$x' \to x'' = x' - ut', t' \to t'' = t',$$
$$f(x',t') \to f''(x'',t'') = -i\left[\frac{1}{2}ux' + \frac{1}{2}\vec{u}\cdot\vec{u}t'\right]f(x',t'),$$

which can also retain the nonlinear Schrödinger equation invariance. For an infinitesimal velocity $u, d\vec{x}' = -ut', dt' = 0$ and $d\hat{f} = f''(x'',t'') - f(x',t') = -(i/2)ux'f(x',t')$. After integration over the space variables, equation (42) leads to the conservation law Eq. (44) which implies that the velocity of the center of mass of the microscopic particles is a constant. It is also the same, even though the particle is in motion. This exhibits clearly that the microscopic particles have the particulate nature.

3.5. The Nonlinear Quantum Mechanics Describes Hamiltonian Systems, the Behavior of which is Determined by a Set of Canonical Conjugate Variables. The States of Particles Can be Described by Lagrangian and Hamilton Equations

Using the above variables, f and f^*,one can determine the Poisson bracket and write further the equations of motion of microscopic particles in the form of Hamilton's equations. For Eq. (5) with $V(\vec{r},t) = A(f) = 0$, the variables f and f^* satisfy the Poisson bracket [34,27]:

$$\left\{f^{(a)}(x), f^{(b)}(y)\right\} = id^{ab}d(x-y) \qquad (45)$$

where $\{A,B\} = i\int_{-\infty}^{\infty}\left(\frac{dA}{df}\frac{dB}{df^*} - \frac{dB}{df}\frac{dA}{df^*}\right)$

The corresponding Lagrangian density L in Eq. (16) associated with Eq. (5) with $A(f) = 0$ can be written in terms of $f(x,t)$ and its conjugate f^* viewed as independent variables. The action of the system is the functions of $f, \nabla f, f_{t'}$ and $f^*, \nabla f^*, f^*_{t'}$, and is represented by Eq.(37). In accordance with the theorem of variation, the variation of the

action for infinitesimal df and df^* is of the form

$$dA=\int_{t_B}^{t'}\int_D\left[\frac{\partial L'}{\partial f}df+\frac{\partial L'}{\partial \nabla f}d\nabla f+\frac{\partial L'}{\partial f_{t'}}df_{t'}\right]\mathrm{d}x'\,\mathrm{d}t'+\mathrm{c.c.} \tag{46}$$

where L' = L, $\partial L'/\partial(\nabla f)$ denotes the vector with components $\partial L'/\partial(\partial_i f)(i=1,2,3)$. After integrating by parts, we get

$$dA=\int_{t_b}^{t'}\int_D\left[\frac{\partial L'}{\partial f}-\nabla\cdot\left(\frac{\partial L'}{\partial \nabla f}\right)-\partial_{t'}\left(\frac{\partial L'}{\partial f_{t'}}\right)\right]df\,dx'dt'+\left[\frac{\partial L'}{\partial f_t}df\right]_{t_0}^{t_1}+\mathrm{c.c.} \tag{47}$$

A necessary and sufficient condition for a function $f(x',t')$ with known values $f(x',t_0')$ and $f(x',t_1')$ to yield an extremum of the action A is that it must satisfy the Euler-Lagrange equation

$$\frac{\partial L'}{\partial f}=\nabla\cdot\left(\frac{\partial L'}{\partial \nabla f}\right)+\partial_{t'}\left(\frac{\partial L'}{\partial f_{t'}}\right) \tag{48}$$

Eq.(48) gives the nonlinear Schrödinger equation (5) if the Lagrangian density Eq. (16) is used. Therefore, the dynamic equation, or the nonlinear *Schrödinger* equation in nonlinear quantum mechanics can be derived from the Euler-Lagrange equation, if the Lagrangian function of the system is known. This is different from linear quantum mechanics, in which a dynamic equation, or the linear Schrödinger equation, cannot be obtained from the Euler-Lagrange equation. This is a unique property of nonlinear quantum mechanics.

The above derivation of the nonlinear *Schrödinger* equation based on the variational principle is a foundation for other methods such as the "the collective coordinates", the "variational approach", where a solution is assumed to maintain a prescribed approximate profile (often bell-type) [26-27]. Such methods greatly simplify the problem, reducing it to a system of ordinary differential equations for the evolution of a few characteristics of the systems. Therefore, this method is extensively used.

On the other hand, the Hamilton equation can also obtain from the Hamiltonian density of this system in Eq.(17). In fact, we can obtain from Eq.(17) [26-29]

$$\frac{dH'}{df^*}=-\frac{\eta^2}{2m}\nabla^2 f+V(x)f-b(f*f)f$$

Where H' = H. . Then from Eq.(5) $A(f)=0$ we can give

$$i\eta\frac{\partial f}{\partial t}=\frac{dH'}{df^*}=-\frac{\eta^2}{2m}\nabla^2 f+V(x)f-b(f^*f)f$$

Thus $i\eta\frac{\partial f}{\partial t}=\frac{dH'}{df^*}$, or $i\eta\frac{\partial f^*}{\partial t}=-\frac{dH'}{df}$ (49)

Equation (49) is just the complex form of Hamilton equation. This form can also represent as normal form denoted by canonical coordinate and momentum of the particle. Thus we have to introduce the following canonical coordinate and momentum:

$$q_1=\frac{1}{2}\left(f+f^*\right),\quad p_1=\frac{\partial L'}{\partial\left(\partial_{t'}q_1\right)};q_2=\frac{1}{2i}\left(f-f^*\right),\quad p_2=\frac{\partial L'}{\partial\left(\partial_{t'}q_2\right)}$$

Thus, the Hamiltonian density of the system in Eq.(17) takes the form

$$\mathsf{H}=\sum_i p_i\partial_{t'}q_i-\mathsf{L}$$

and the corresponding variation of the Lagrangian density L = L' can be written as

$$\delta L'=\sum_i\frac{dL'}{dq_i}dq_i+\frac{dL'}{d\left(\nabla q_i\right)}d\left(\nabla q_i\right)+\frac{dL'}{d\left(\partial_{t'}q_i\right)}d\left(\partial_{t'}q_i\right)\qquad(50)$$

From Eq.(50), the definition of p_i, and the Euler-Lagrange equation in such a case,

$$\frac{\partial L'}{\partial q_i}=\nabla.\frac{\partial L'}{\partial\nabla q_i}+\frac{\partial p_i}{\partial t}$$

one obtains the variation of the Hamiltonian in the form of

$$\delta\mathsf{H}=\sum_i\int\left(\partial_{t'}q_i dp_i-\partial_{t'}p_i dq_i\right)dx'$$

Thus, the canonical form of Hamilton equation can be derived:

$$\frac{\partial q_i}{\partial t'}=\delta\mathsf{H}/\delta\mathsf{p}_i\ ,\quad\frac{\partial p_i}{\partial t'}=-\delta\mathsf{H}/\delta\mathsf{q}_i\qquad(51)$$

This is also interesting. It shows that the nonlinear *Schrödinger* equation of dynamics describing microscopic particle can be obtained from the classical Hamilton equation in nonlinear quantum mechanics, if the Hamiltonian of the system is known. Obviously, such methods of finding dynamic equations are impossible in the linear quantum mechanics. As is known, the Euler-Lagrange equation and Hamilton equation are important equations in classical theoretical (analytic) mechanics, and were used to describe laws of motions of classical particles. These equations are now used to depict properties of motions of microscopic particles in nonlinear quantum mechanics. This shows sufficiently the classical features of microscopic particles in nonlinear quantum mechanics. On the other hand, from

this study, we seek new ways of finding the equation of motion of the microscopic particles in nonlinear quantum mechanics, i.e., if the Lagrangian or Hamiltonian of the system is known in the coordinate representation, then we can obtain the equation of motion of MIP from the Euler-Lagrange or Hamilton equations[34, 26-27].

3.6. The Motion of Particles Obeys a Classical Rule of Motion in Nonlinear Quantum Mechanics

Now utilizing Eq. (5) with $A(f)=0$ and its conjugate equation we can obtain[12, 26-27,35]

$$\frac{d}{dt'}\int_{-\infty}^{\infty} f^* f_{x'}dx' = \int_{-\infty}^{\infty} f_{t'}^* f_{x'}dx' + \int_{-\infty}^{\infty} f^*(f_{t'})_{x'}dx' = i\int_{-\infty}^{\infty}\{f^*\frac{\partial}{\partial x'}[f_{x'x'} + bf^* f^2$$

$$-Vf] - [f_{x'x'}^* - bf(f^*)^2 - Vf^*]f_{x'}\}dx' = i\int_{-\infty}^{\infty} f^*\frac{\partial V}{\partial x'}f\,dx'$$

We here utilize the following relations and the boundary conditions:

$$\int_{-\infty}^{\infty}(f^* f_{x'x'x'} - f_{x'x'}^* f_{x'})dx' = 0, \int_{-\infty}^{\infty} b(f^{*2} f f_{x'} + f^* f^2 f_{x'}^*)dx' = 0$$

$$\lim_{|x'|\to\infty} f(x',t') = \lim_{|x'|\to\infty} f_{x'}(x',t') = 0 \text{ and } \int_{-\infty}^{\infty} f^* f\,dx' = const. \lim_{|x'|\to\infty} f^* x' f_{x'} = \lim_{|x'|\to\infty} f_{x'}^* x' f = 0$$

where $f_{x'} = \frac{\partial f}{\partial x'}, f_{x'x'x'} = \frac{\partial^3 f}{\partial x'^3}$. Thus, we can get

$$\frac{d}{dt'}\int_{-\infty}^{\infty} f^* x' f\,dx' = \int_{-\infty}^{\infty}(\frac{\partial f^*}{\partial t'}x' f + f^* x'(\frac{\partial f}{\partial t'}))dx' = -2i\int_{-\infty}^{\infty} f^* f_{x'}dx'$$

According to the above definition, in the systems, the position of mass centre of microscopic particle is represented by $< x' >= \int_{-\infty}^{\infty} f^* x' f\,dx' / \int_{-\infty}^{\infty} f^* f\,dx'$, the velocity of mass centre of microscopic particle is also denoted by

$$\frac{d}{dt'} < x' >= \frac{\partial}{\partial t'}\{\int_{-\infty}^{\infty} f^* x' f\,dx' / \int_{-\infty}^{\infty} f^* f\,dx'\} = -2\int_{-\infty}^{\infty} j^* j_{x'}dx' / \int_{-\infty}^{\infty} f^* f\,dx'$$

Thus, the acceleration of mass centre of microscopic particle can also be denoted by

$$\frac{d^2}{dt'^2} < x' >= -2i\frac{d}{dt'}\{\int_{-\infty}^{\infty} f^* f_{x'}dx' / \int_{-\infty}^{\infty} f^* f\,dx'\} = -2i\int_{-\infty}^{\infty} f^* V_{x'} f^* dx' = -2 < \frac{\partial V}{\partial x'} > \quad (52)$$

If f is normalized, i.e., $\int_{-\infty}^{\infty} f^* f dx' = 1$, then the above conclusions also are not changed.

However,generally speaking, $\frac{\partial V(<x'>)}{\partial <x'>} \neq <\frac{\partial V(x')}{\partial x'}>$, then we have to expand $\frac{\partial V}{\partial x'}$ at $x' = <x'>$ by

$$\frac{\partial V(x')}{\partial x'} = \frac{\partial V(<x'>)}{\partial <x'>} + (x' - <x'>)\frac{\partial^2 V(<x'>)}{\partial <x'>^2} + \frac{1}{2!}(x' - <x'>)^2 \frac{\partial^3 V(<x'>)}{\partial <x'>^3} + \mathrm{L}$$

Finding the expectation value to the above formula, thus we get

$$\left\langle \frac{\partial V(x')}{\partial x'} \right\rangle = \frac{\partial V(<x'>)}{\partial <x'>} + \frac{1}{2!} <(x' - <x'>)^2> \frac{\partial^3 V(<x'>)}{\partial <x'>^3}$$

For the microscopic particle, describing by Eq.(9), the position of the particle, that is, the position of the mass center of the particle is known, is just $<x'> = x_0'$ =constant, or 0 .When the motion of the mass centre of the particles is only studied, we may calculate the value related to the mass centre in finding the value of $<(x' - <x'>)^2>$.Thus we can obtain $<(x' - <x'>)^2> = 0$. Hence, $\left\langle \frac{\partial V(x')}{\partial x'} \right\rangle = \frac{\partial V(<x'>)}{\partial <x'>}$. Finally, we can get

$$\frac{d^2}{dt'^2} <x'> = -2\frac{\partial V(<x'>)}{\partial <x'>} \quad \text{or} \quad m\frac{d^2 x_0}{dt^2} = -\frac{\partial V}{\partial x_0} \tag{53}$$

where $x_0 = <x>$ is the position of the mass centre of MIP. Equation (53) is a Newton-type classical equation of motion. This shows clearly that the motion of the mass centre of MIP satisfies the Newton law in nonlinear quantum mechanics[28-27]. Therefore, we can say that the microscopic particle has the property of the classical particle. If $V =$ constants in Eq.(5) with $A(f) = 0$ we can get from Eqs.(52)-(53) that $m\frac{d^2}{dt'^2} <x'> = -2\frac{\partial V(<x'>)}{\partial <x'>} = 0$. This shows that the MIP moves in uniform velocity in space-time. For the solution Eq.(9) we can get that the acceleration of the mass centre of MIP is just zero because $V = 0$. Therefore, the velocity of the particle is a constant. In fact, if we insert Eq. (9) into Eq.(53) we can obtain $v_g = d<x'>/dt' = V_e$=constant. This shows clearly that the velocity of the uniform motion of MIP is just the group velocity of the soliton. This property of the motion of microscopic particle shows that its energy and momentum can be retained in the motion process.

The above equation of motion of microscopic particles can also be derived from the nonlinear *Schrödinger* equation (5) with $A(f) = 0$ by means of another method. As is

known, the energy E and momentum of MIP described by the nonlinear *Schrödinger* equation (5) with $A(f)=0$ are denoted by Eq.(18) and $P=-i\int_{-\infty}^{\infty}\left(f^{*}f_{x'}-f_{x'}^{*}f\right)dx'$, respectively. For this system, the energy E and quantum number $N_s=\int_{-\infty}^{\infty}|f|^2\,dx'$ are integral invariant. However, the momentum P is not conserved and has , from the above result, the following property:

$$\frac{dP}{dt'}=\int_{-\infty}^{\infty}2V(x')\frac{\partial}{\partial x'}|f|^2\,dx'=-2\int_{-\infty}^{\infty}\frac{\partial V}{\partial x'}|f|^2\,dx' \tag{54}$$

where the boundary condition is $f(x')\to 0$ as $|x'|\to\infty$. For slowly varying inhomogeneities (in comparison with particle scale (soliton)), *i.e.* W_s>>L, where L is the inhomogeneity scale, W_s is its width, expanding Eq. (54) into a power series in W_s/L and keeping only the leading term, we can get

$$\frac{dP}{dt'}=-2\frac{\partial V(x_0')}{\partial x_0'}N_s \tag{55}$$

where x_0' is the position of the center of the mass of the macroscopic particle. Eq. (54) or (55) is essentially consistent with Eqs.(52)−(53) which are in the form of the equation of motion for a classical particle. Indeed, if we write the particle (soliton) solution as

$$f(x',t')=j\,(x'-x_0',t')\,e^{ip(x'-x_0')+iq} \tag{56}$$

we assume that it is a solution of Eq. (5) at $A(f)=0$. Inserting Eq. (56) into the representation of *P*(*x*,*t*), we get $P=pN_s$. Let $p=\frac{dx_0'}{dt'}$ be the velocity of the center of the particle, then equation (55) and $P=pN_s$ indicate that the center of mass of the microscopic particle moves like a classical particle in a weakly inhomogeneous potential field $V(x_0')$ according to

$$\frac{d^2x_0'}{dt'^2}=-2\frac{\partial V}{\partial x_0'}\ \text{ or }\ m\frac{d^2x_0}{dt^2}=-\frac{\partial V}{\partial x_0} \tag{57}$$

This is the same as Eq. (53) and it is Newton's equation for a classical particle.

3.7. Features of Motion of Microscopic Particles in Several Special Potentials

We now consider some particular cases. Let $V(x')=a\,x'$ in Eq.(5) with $A(f)=0$, where a is constant, and make the following transformation [36-37, 26-27]:

$$f(x',t')=f'(x',t')\,e^{-ia\,x't'+ia^2t'^3/3},\quad x'=x-a\,t',\quad t'=t,$$

Then Eq. (5) with $A(f)=0$ becomes

$$if'_{t'}+f'_{x'x'}+2|f'|^2f'=0, \tag{58}$$

where $b=2$. The exact and complete solution of the above equation is well known. We can thus obtain the complete solution of the nonlinear *Schrödinger* equation with $V(x')=ax'$. Its single soliton solution is given by

$$\begin{aligned} f=&2h\,\mathrm{sech}\left[2h\left(x'-4xt'+2at'^2-x'_0\right)\right]\times\\ &\exp\left\{i\left[2(x-at')x'+\frac{4a^2t'^3}{3}-4axt'^2+4\left(x^2-h^2\right)t'+q_0\right]\right\}\end{aligned} \tag{59}$$

When $V_0(x')=a^2x'^2$, we can get

$$\begin{aligned} f=&2h\,sec\,h\left\{2h(x'-x'_0-\frac{4xh}{a}sin\left[2a\,(t'-t'_0)\right]\right\}\times\\ &exp\left\{i\left[2x(x'-x'_0-cos\,2a\,(t'-t'_0)-\frac{x^2}{a}sin\left[4a\,(t'-t'_0)\right]+4h^2(t'-t'_0)+q'_0\right]\right\}\end{aligned} \tag{60}$$

In each of the above two cases, with two different external potential fields, the characteristics of motion of the microscopic particle can be determined according to Eq. (57). The accelerations of the center of mass of the microscopic particle is given by

$$\frac{\mathrm{d}^2x'_0}{\mathrm{d}t'^2}=-2\frac{\partial\mathrm{V}\left(\langle\mathrm{x}'\rangle\right)}{\partial\langle\mathrm{x}'\rangle}=-2a=\text{constant} \tag{61}$$

for $V(x')=a\,x'$,and

$$\frac{d^2x_0'}{dt'^2} = -4a^2x_0' \tag{62}$$

for $V(x') = a^2x'^2$ [36-37, 26-27], respectively.

These results can also be obtained using the following method. From de Broglie relation $E = hu = \hbar w$ and $\vec{P} = \hbar\vec{k}$ for microscopic particles which represent the wave-corpuscle duality in quantum theory, the frequency w retains its role as the Hamiltonian of the system even in this complicated and nonlinear systems and

$$\frac{dw}{dt'} = \frac{\partial w}{\partial k}\bigg|_{x'}\frac{dk}{dt} + \frac{\partial w}{\partial x'}\bigg|_{k}\frac{\partial x'}{\partial t'} = 0$$

as in the usual stationary media[26-27]. From the above result we also know that the usual Hamilton equation in Eq. (34) for the nonlinear quantum mechanical systems remain valid for the microscopic particles. Thus, the Hamilton equation in Eq. (34) can be now represented by another form:

$$\frac{dk}{dt'} = -\frac{\partial w}{\partial x'}\bigg|_{k}, \frac{dx'}{dt'} = \frac{\partial w}{\partial k}\bigg|_{x'} \tag{63}$$

in the energy picture, where $k = \partial q/\partial x'$ is the time-dependent wave number of the microscopic particle, $w = -\partial q/\partial t'$ is its frequency , q is the phase of the wave function of the microscopic particles. From Eqs. (59) and (60), we know that

$$q = 2(x - at')x' + \frac{4a^2t'^3}{3} - 4axt'^2 + 4(x^2 - h^2)t' + q_0,$$

for $V(x') = ax'$ and

$$q = 2z\,x'\cos 2a(t' - t_0') + \left(\frac{z^2}{a}\right)\sin 4a(t' - t_0') + 4h^2(t' - t_0') + q_0',$$

for $V(x') = a^2x'^2$, respectively. From Eq.(59) we can find that for $V(x') = ax'$,

$$k = 2(x - at'),$$
$$w = 2ax' - 4(2x - at')^2 + (2h)^2 = 2ax' - k^2 + (2h)^2.$$

Thus, the group velocity of the microscopic particle is

$$u_{\mathrm{g}}=\frac{dx'}{dt'}=\left.\frac{\partial w}{\partial k}\right|_{x'}=4\left(2x-at'\right),$$

and its acceleration is given by

$$\frac{d^2x'}{dt'^2}=\frac{dk}{dt'}=-2a=\text{const ant},\ \ \text{here}(x_0'=x_0') \tag{64}$$

For Eq. (60) , we have

$$V\left(x'\right)=a^2x'^2,k=2x\cos 2a\left(t'-t_0'\right),$$
$$w=4axx'\sin 2a\left(t'-t_0'\right)-4x^2\cos 4a\left(t'-t_0'\right)-4h^2$$
$$=2a\,x'\left(4x^2-k^2\right)^{1/2}-2k^2+4\left(x^2-h^2\right),$$

Thus, the group velocity of the microscopic particle is

$$v_{\mathrm{g}}=\left.\frac{\partial w}{\partial k}\right|_{x'}=\frac{a\,x'}{x}\frac{k}{\sqrt{1-k^2/4x^2}}-2k=2a\,x'\,\mathrm{ctg}\left[2a\left(t'-t_0'\right)\right]-4x\cos\left[2a\left(t'-t_0'\right)\right],$$

while its acceleration is

$$\frac{dk}{dt'}=-\left.\frac{\partial w}{\partial x'}\right|_{k}=-2a\sqrt{4x^2-k^2}=-4x\,a\,\sin\left[2a\left(t'-t_0'\right)\right].$$

since $\dfrac{d^2x'}{dt'^2}=\dfrac{dk}{dt'}$, here $\left(x'=x_0'\right)$.

We have

$$\frac{dk}{dt'}=\frac{d^2x'}{dt'^2}=-4z\,a\,\sin\left[2a\left(t'-t_0'\right)\right],$$

and

$$x'=\frac{2x}{a}\sin\left[2a\left(t'-t_0'\right)\right].$$

Finally, the acceleration of the microscopic particle is

$$\frac{d^2\overline{x}'}{dt'} = \frac{dp^k}{dt'} = -4a^2\overline{x}'. \tag{65}$$

Eqs. (64) and (65) are exactly the same as Eqs.(61) and (62), respectively, which shows that Eqs. (53) or (55)and (57) have the same effects and function as Eqs.(51) and (63) in nonlinear quantum mechanics. On the other hand, it is well known that a macroscopic object moves with a uniform acceleration, when $V(x') = ax'$ which corresponds to the motion of a charge particle in a uniform electric field, and when $V(x') = a^2x'^2$ which is a harmonic potential, the macroscopic object performs localized vibration with a frequency of $2a$ and an amplitude of $2x/a$, and the corresponding classical vibrational equation is $x' = x_0' \sin wt'$, with $w = 2a$ *and* $x_0' = x/a$. The equations of motion of the macroscopic object are consistent with Eq. (57) and Eqs. (61) – (62) or (64)-(65) for the center of mass of microscopic particles in the nonlinear quantum mechanics. These correspondence between a microscopic particle and a macroscopic object shows[26-27] that microscopic particles in nonlinear quantum mechanics have exactly the same properties as classical particles, and their motion satisfy the classical laws of motion. We have thus demonstrated clearly from the dynamic equations (nonlinear Schrodinger equation), the Hamiltonian or Lagrangian of the systems, and the solutions of equations of motion, in both uniform and inhomogeneous systems, that microscopic particles in nonlinear quantum mechanics really have the corpuscle property.

3.8. Mechanism and Rules of Collision of the Microscopic Particle

As is known, the most obvious feature of macroscopic particles is meeting the collision law or conservation law of momentum. Therefore, we often also use the law to determine the particulate feature of macroscopic particles. In Figure 1(d), we show also the collision feature by numerical simulation method for the solution of the nonlinear Schrodinger equation(NLSE) (5). From this figure, we see that microscopic particle satisfies the collision law of macroscopic particles. As a matter of fact, Zakharov *et al.*[31] demonstrated that the solutions of Eq.(5) with V(x,t)= $A(f) = 0$ obey also the collision law of macroscopic particles by theoretical analysis at both b>0 and b<0. Their results show that when microscopic particles collide with other particles, the faster particle moves forward by an amount of phase shift, and the slower one shifts backwards by an amount of phase. The total shift of the particles is equal to the algebraic sum of those of the pair during the paired collisions. At the same time, experiments and numerical simulations also show during the collision that the MIPs interact and exchange positions in the space-time trajectory as if they had passed through each other.

After the collision, the two MIPs may appear to be instantly translated in space and/or time but otherwise unaffected by their interaction. The translation is called a phase shift as mentioned above. In one dimension, this process results from two MIPs colliding head-on from opposite directions, or in one direction between two particles with different amplitudes. This is possible because the velocity of a particle depends on the amplitude. The two MIPs surviving a collision completely unscathed demonstrates clearly the corpuscle feature of the microscopic particles. This property separates nonlinear quantum mechanical microscopic particles (solitons) from particles in the linear quantum mechanical regime. Therefore, the rule of collision of MIPs is the same as that of classical particles.

In the following, we describe a series of laboratory and numerical experiments dedicated to investigate the detailed structure, mechanism and rules of collision between the microscopic particles described by the nonlinear Schrödinger equation in nonlinear quantum mechanics. The properties and rules of such collision between two solitons of NLSE (5) at V(x,t)= $A(f)=0$ have been first studied by Aossey *et al.*[38]. Both the phase shift of the microscopic particles after their interaction and the range of the interaction are functions of the relative amplitude of the two colliding solitons. The solitons preserve the shape after the collision. We here discuss the rule of collision of two MIPs depicted by nonlinear Schrodinger equation(5) at V(x,t)= $A(f)=0$ by means of Aossey et al's method and results.

For the microscopic particles described by the nonlinear Schrödinger equation (5) with V(x,t)= $A(f)=0$, we will limit our discussion to the hole (dark) spatial particles (solitons) with $b<0$ [26-27]. Aossey et al now denoted the hole-particle by

$$f(x',t')=f_0\sqrt{1-B^2\,\mathrm{sech}^2(x')}e^{\pm i\Theta(x')} \tag{66}$$

where

$$\Theta(x')=\sin^{-1}\left[\frac{B\tanh(x')}{\sqrt{1-B^2\,\mathrm{sec\,h}^2(x')}}\right],x'=m\left(x'-x_0'-u_t t'\right)$$

Here, B is a measure of the amplitude ("blackness") of the solitary wave (hole or dark soliton) and can take a value between -1 and 1, u_t is the dimensionless transverse velocity of the particle center, and m is the shape factor of the particle. The intensity (I_d) of the solitary wave (or the depth of the irradiance minimum of the dark soliton) is given by $B^2f_0^2$. Aossey *et al.* showed that the shape factor m and the transverse velocity u_t are related to the amplitude of the particles, which can be obtained from the nonlinear *Schrödinger* equation in the optical fiber to be

$$m^2 = n_0 |n_2| m_0^2 B^2 f_0^2, u_t \approx \pm\sqrt{(1-B^2)\frac{|n_2| f_0^2}{n_0}}$$

where n_0 and n_2 are the linear and nonlinear indices of refraction for the optical fiber material. We have assumed $|n_2| f_0^2 = n_0$. When two microscopic particles (solitons) described by NLSE collide, their individual phase shifts are given by

$$d x_j = \sqrt{\frac{n_0}{|n_2| f_0^2}} \frac{1}{2 m_0 n_0 B_j} \ln\left[\frac{\left(\sqrt{1-B_1^2}+\sqrt{1-B_2^2}\right)^2 + (B_1+B_2)^2}{\left(\sqrt{1-B_1^2}+\sqrt{1-B_2^2}\right)^2 + (B_1-B_2)^2}\right] \tag{67}$$

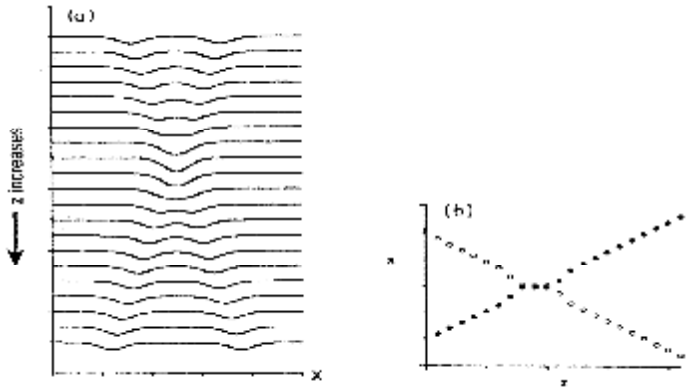

Figure 2 Numerical simulation of an overtaking collision of equi-amplitude dark solitons. (a) Sequence of the waves at equal intervals in the longitudinal position z. (b) Time-of-flight diagrams of the signal.

The MIP (soliton) interaction can be easily investigated numerically by using a split-step propagation algorithm which was found, by Thusrston *et al.*[39], to closely predict experimental results. The results of a simulated collision between two equi-amplitude microscopic particles (solitons) are shown in Figure2 (a), which are similar to that of general MIPs (bright solitons) depicted by Eq.(5) at V(x,t)= $A(f)=0$ and b>0 as shown in Figure1$_d$. We note that the two particles interpenetrate each other, retain their shape, energy and momentum, but experience a phase shift at the point of collision. In addition, there is also a well- defined interaction length in z along the axis of time t that depends on the relative amplitude of two colliding MIPs. This case occurs also in the collision of two KdV solitons. Cooney *et al.*[40] studied the overtaking collision, to verify the KdV soliton nature of an observed signal in the plasma experiment. In the following, we discuss a fairly simple model which was used to simulate and to interpret the experimental results on the MIPs (solitons) described by NLSE and KdV solitons.

The model is based on the fundamental property of solitons that two MIPs (solitons) can interact and collide, but survive the collision and remain unchanged. Rather than using the exact functional form of $\mathrm{sech}x$ for MIPs (solitons) described by NLSE, the MIPs are

represented by rectangular pulses with an amplitude A_j and a width W_j where the subscript j denotes the j th microscopic particles. An evolution of the collision of two MIPs is shown in Figure3(a) by Aossey et al.[38] . In this case, Aossey *et al.* considered two solitons(MIPs) with different amplitudes. The details of what occurs during the collision need not concern us here other than to note that the MIPs with the larger-amplitude has completely passed through the one with the smaller amplitude. In regions which can be considered external to the collision, the MIPs do not overlap as there is no longer an interaction between them. The microscopic particles are separated by a distance, $D = D_1 + D_2$, after the interaction. This manifests itself in a phase shift in the trajectories depicted in Figure 3(b). This was noted in the experimental and numerical results. The minimum distance is given by the half-widths of the two microscopic particles, $D \geq W_1/2 + W_2/2$. Therefore,

$$D_1 \; ? \; \frac{W_1}{2} \text{ and } D_2 \; ? \; \frac{W_2}{2} \tag{68}$$

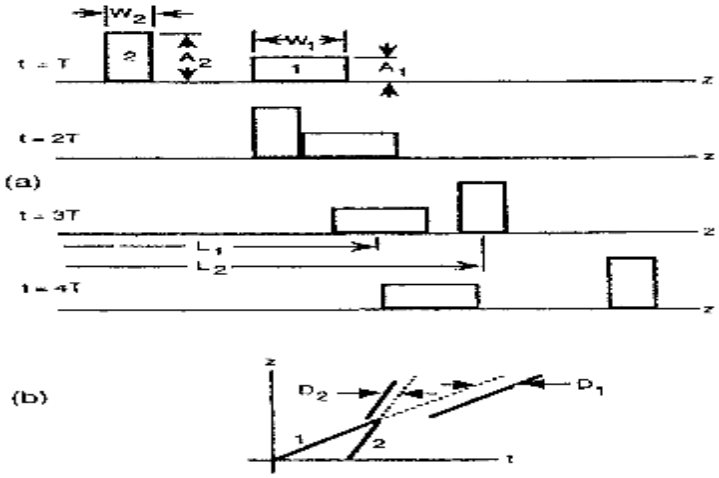

Figure 3 Overtaking collision of solitons. (a) Model of the interaction just prior to the collision and just after the collision. After the collision, the two MIPs are shifted in phase. (b) Time-of-light diagram of the signals. The phase shifts are indicated.

Another property of the microscopic particles (solitons) is that their amplitude and width are related. For the microscopic particles described by the nonlinear *Schrödinger* equation with $b < 0$ in Eq. (5) of V(x,t)= $A(f) = 0$ ($W \approx 1/m$), Aossey et al obtained

$$B_j W_j = \text{constant} = K_1 \tag{69}$$

Using the minimum values in Eq.(68), we find that the ratio of the repulsive shifts for the microscopic particles described by the nonlinear *Schrödinger* equation is given by

$$\frac{D_1}{D_2}=\frac{B_2}{B_1} \tag{70}$$

Results obtained from simulation of the kind of solitons are presented in Figure 4(a). The solid line in the figure corresponds to Eq.(70).\

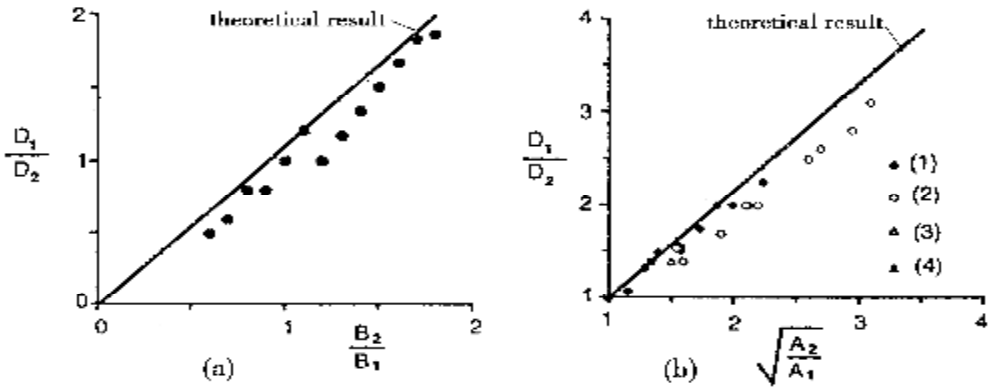

Figure 4 Summary of the ratio of the measured phase shifts as a function of the ratio of amplitudes. (a) For the solitons described by NLSE, the solid line corresponds to Eq.(67). (b) KdV solitons, the data are from (1) this experiment, (2) Zabusky et al. [31], (3) Lamb's[41] and (4) Ikezi et al.'s [42] results. The solid line corresponds to Eq.(74).

In addition to predicting the phase shift that results from the collision of two microscopic particles, the model also allows us to estimate the size of the collision region or duration of the collision. Each microscopic particles depicted in Figure3 travels with its own amplitude-dependent velocity u_j. For the two microscopic particles to interchange their positions during a time ΔT, they must travel a distance L_1 and L_2,

$$L_1=u_1\Delta T \text{ and } L_2=u_2\Delta T \tag{71}$$

The interaction length must then satisfy the relation

$$L=L_2-L_1=(u_2-u_1)\Delta T \;?\; W_1+W_2 \tag{72}$$

Equation (71) can be written in terms of the amplitudes of the two MIPs. For the MIPs described by NLSE, combining Eqs. (68) and (72), Aossay *et al.* obtained

$$L\geq K_1\left[\frac{1}{B_1}+\frac{1}{B_2}\right] \tag{73}$$

In Figure 5(a), the results for the MIPs described by NLSE are presented. The dashed line corresponds to Eq. (73) with $B_2 = 1$ and $K_1 = 6$. The interaction time (solid line) is the sum of the widths of the two microscopic particles, minus their repulsive phase shifts, and multiplied by the transverse velocity of MIP 1. Since the longitudinal velocity is a constant, this scales as the interaction length. From the figure, we see that the theoretical result obtained using the simple collision model is in good agreement with that of the numerical simulation.

The discussion presented above and the corresponding formulae reveal the mechanism and rule of the collision between MIPs depicted by NLSE in the nonlinear quantum mechanics.

To verify the validity of this simple collision model, Aossey *et al.* studied the collision of the solitons using the exact form of $sec\,h^2 x$ for the KdV equation $u_t + uu_x + d'u_{xxx} = 0$, and the collision model shown in Figure3. For the KdV soliton they found that

$$A_j \left(W_j\right)^2 = \text{cons tan } t = K_2 \text{ and } \frac{D_1}{D_2} = \frac{W_1/2}{W_2/2} = \sqrt{\frac{A_2}{A_1}} \tag{74}$$

where A_j and W_j are the amplitude and width of the j th KdV soliton, respectively. Corresponding to the above, Aossey *et al.* obtained for the interaction length.

$$L \geq K_2 \left(\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} \right) = \frac{K_2}{\sqrt{A_1}} \left(1 + \sqrt{\frac{A_1}{A_2}} \right) \tag{75}$$

Aossey *et al.*[38] compared their results for the ratio of the phase shifts as a function of the ratio of the amplitudes for the KdV solitons, with those obtained in the experiments of Ikezi, Taylor, and Baker[42], and those obtained from numerical work of Zabusky and Kruskal[31] and Lamb[41], as shown in Figure4(b). The solid line in Figure 4(b) corresponds to Eq. (74). Results obtained by Aossey *et al.* for the interaction length are shown in Figure 5(b) as a function of amplitudes of the colliding KdV solitons. Numerical results (which were scaled) from Zabusky and Kruskal are also shown for comparison. The dashed line in Figure 5(b) corresponds to Eq. (75), with $A_1 = 1$ and $K_2 = 1$.

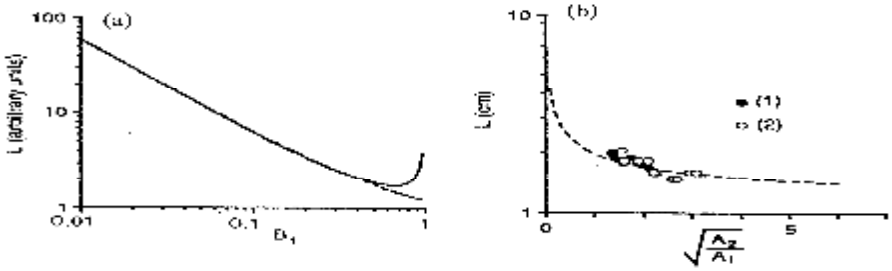

Figure 5. Summary of the measured interaction length as a function of the amplitudes. (a) The particles described by NLSE, the dashed line corresponds to Eq. (70) with $B_2 = 1$ and $K_1 = 6$. (b) KdV solitons, the data are from (1) this experiment and (2) Zabusky et al.'s result [31]. The dashed line corresponds to Eq. (72) with $K_2 = A_1 = 1$.

Since the theoretical results obtained by the collision model based on macroscopic bodies in Figure 3 are consistent with experimental data for the KdV soliton, shown in Figs. 4(b) and 5(b), it is reasonable to believe the validity of the above theoretic results of model of collision presented above , and results shown in Figs.4(a) and 5(a) for the microscopic particles described in the nonlinear Schrödinger equation which are obtained using the same model as that shown in Fig 3. Thus, the above colliding mechanism for the microscopic particles shows clearly the classical corpuscle feature of the microscopic particles in nonlinear quantum mechanics.

3.9. Features of Reflection and Transmission of Microscopic Particles at Interfaces

As mentioned above, microscopic particles in nonlinear quantum mechanics represented by Eq. (5) also have wave property, in addition to the corpuscle property. This wave feature can be conjectured from the following reasons.

1) Eqs. (5)−(8) are wave equations and their solutions, Eqs. (9)−(10) and (15) are solitary waves having the features of traveling waves. A solitary wave consists of a carrier wave and an envelope wave, has certain amplitude, width, velocity, frequency, wavevector, and so on, and satisfies the principles of superposition of waves, although the latter are different when compared with classical waves or the de Broglie waves in linear quantum mechanics.
2) The solitary waves have reflection, transmission, scattering, diffraction and tunneling effects, just as that of classical waves or the de Broglie waves in linear quantum mechanics. At present, we study the reflection and transmission of the microscopic particles at an interface.

The propagation of microscopic particles (solitons) in a nonlinear nonuniform media is different from that in uniform media. The nonuniformity can be due to a physical confining structure or two nonlinear materials being juxtaposed. One could expect that a portion of microscopic particles that was incident upon such an interface from one side would be reflected and a portion would be transmitted to the other side due to its wave feature. Lonngren *et al.* [43] observed the reflection and transmission of microscopic particles (solitons) in a plasma consisting of a positive ion and a negative ion interface, and numerically simulated the phenomena at the interface of two nonlinear materials. To illustrate the rules of reflection and transmission of microscopic particles, we discuss here the work of Lonngren *et al.* [43]

Lonngren *et al.* [41] simulated numerically the behaviors of solitons (MIPs) described by

NLSE (5) at V(x,t)= $A(f)=0$ They found that the signal had the property of a soliton.These results are in agreement with numerical investigations of similar problems by Aceves et al. A sequence of pictures obtained by Lonngren *et al.* [41] at uniform temporal increments of the spatial evolution of the signal are shown in Figure 6. From this figure, we note that the incident microscopic particles propagating toward the interface between the two nonlinear media splits into a reflected and transmitted soliton at the interface. From the numerical values used in producing the figure, the relative amplitudes of the incident, the reflected and the transmitted solitons can be deduced.

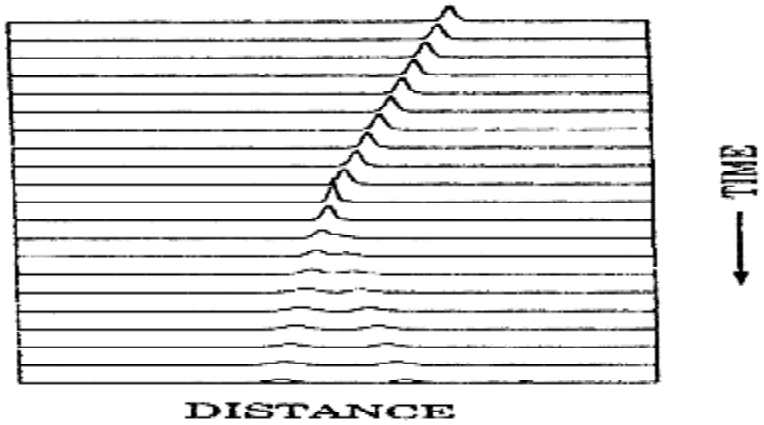

Figure 6. Simulation results showing the collision and scattering of an incident solitons described by NLSE (top) onto an interface. The peak nonlinear refractive index change is 0.67% of the linear refractive index for the incident solitons and the linear offset between the two regions is also 0.67%.

They assumed that the energy that is carried by the incident soliton (MIP) is all transferred to either the transmitted or the reflected solitons and none is lost through radiation. Thus

$$E_{\text{inc}} = E_{\text{ref}} + E_{\text{trans}}$$

Lonngren *et al.* gave approximately the energy of soliton(MIP) by

$$E_j = \frac{A_j^2}{Z_c} W_j ,$$

where the subscript j refers to the incident, reflected or transmitted solitons(MIPs). The amplitude of the soliton(MIP) is A_j and its width is W_j. The characteristic impedance of a material is given by Z_c. Hence, the relation of energy mentioned above can be written as

$$\frac{A_{\text{inc}}^2}{Z_{\text{cI}}} W_{\text{inc}} = \frac{A_{\text{ref}}^2}{Z_{\text{cI}}} W_{\text{ref}} + \frac{A_{\text{trans}}^2}{Z_{\text{cII}}} W_{\text{trans}} \qquad (76)$$

Since $A_j W_j = \text{constant}$ for the soliton described NLSE (see Eq. (69) in which B_j is represented by A_j), we obtain the following relation between the reflection coefficient $R=A_{ref}/A_{inc}$ and the transmission coefficient $T = A_{trans}/A_{inc}$

$$1 = R + \frac{Z_{cI}}{Z_{cII}} T \tag{77}$$

for the MIPs described by NLSE (5) at V(x,t)= $A(f) = 0$. The relation of T versus R is shown in Fig.7.

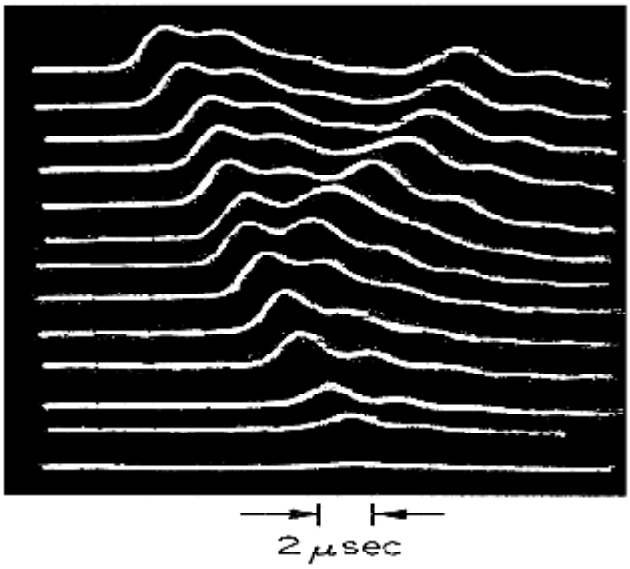

Figure 7. Sequence of the signals detected as the probe is moved in 2 mm increments from 30 to 6 mm in front of the reflector. The incident and reflected KdV solitons coalesce at the point of reflection, which is approximately 16 mm in front of the reflector. A transmitted soliton is observed closer to the disc. The amplitude scale at 8 and 6 mm is increased by 2 from the previous traces.

To verify further this idea, Lonngrel *et al.*[44] conducted experiments with KdV soliton. They found that the detected signal had the characteristics of a KdV soliton. Lonngrel *et al.*[44] showed a sequence of pictures taken using a small probe at equal spatial increments starting initially in a homogeneous plasma sheath adjacent to a perturbing biased object, as shown in Figure7. From this figure, we see that the probe first detects the incident soliton and some time later the reflected soliton. The signals are observed, as expected, to coalesce together as the probe passed through the point where the soliton was actually reflected. Beyond this point which was at the location where the density started to decrease in the steady-state sheath, a transmitted soliton was observed. From Figure7, the relative amplitudes of incident, the reflected and the transmitted solitons can be deduced, which was done by the author.

For the KdV solitons, there is also $A_j W_j^2 = \text{constant}$ (see Eq.(74)) . Difference from Eq.977) of NSE soliton, for the KdVsoliton, Lonngrel et al obtained

$$1 = R^{3/2} + \frac{Z_{eI}}{Z_{eII}} T^{3/2}$$

The relations between the reflection and the transmission coefficients for the soliton(MIP) described by NLSE and KdV soliton are shown in Figure8, with the ratio of characteristic impedances set to one. The experimental results on KdV solitons and results of the numerical simulation of MIPs described by NLSE are also given in this figure. The computed data are shown using triangles. Good agreement between the analytic results and simulation results can be seen. The oscillatory deviation from the analytic result is due to the presence of radiation modes in addition to the soliton modes. The interference between these two types of modes results in the oscillation in the soliton amplitude. In the asymptotic limit, the radiation will spread and damp the oscillation, and result in the reflection –transmission coefficient curve falling on the analytic curve.

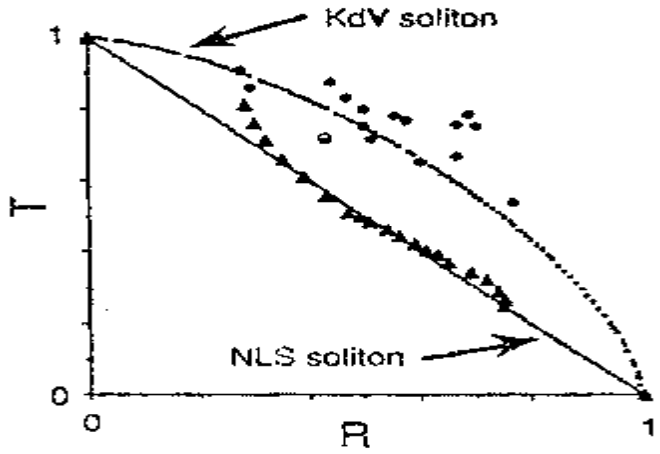

Figure 8. The relationship between the reflection and transmission coefficients of a microscopic particle (soliton)given in Eq.(78). The solid circles are results from the laboratory experiment on KdV solitons and the hollow circle is Y. Nishida's result. The solid triangles are Lonngren et al.'s numerical results for the particle (soliton) described by NLSE.

Figure 8. The relationship between the reflection and transmission coefficients of a NLSsoliton(MIP) given inEqs(77) and (78) are shown in solid line and dashline, respectively. The solid circles are results from the laboratory experiment on KdV solitons and the hollow circle is Y. Nishida's result. The solid triangles are Lonngren et al.'s numerical results for the particle (soliton) described by NLSE.

The above rule of propagation of the microscopic particles in nonlinear quantum mechanics is different from that of linear waves in classical physics. Lonngren *et al.*[45] found that a linear wave obeyed the following relation:

$$1 = R^2 + \frac{Z_{eI}}{Z_{eII}} T^2 \qquad (79)$$

This can be also derived from Eq.(76), by assuming the linear waves. The width of the incident, reflected and transmitted pulses W_j will be the same. For the linear waves

$$R=\frac{Z_{cII}-Z_{cI}}{Z_{cII}+Z_{cI}}, \text{and } T=\frac{2Z_{cII}}{Z_{cII}+Z_{cI}}$$

Thus, Eq. (79) is satisfied. Obviously, Eq. (79) is different from Eq. (77). This shows clearly that the microscopic particles in nonlinear quantum mechanics have a wave feature, but it is different from that of not only linear classical waves and the de Broglie waves in linear quantum mechanics but also KdV solitary wave.

3.10. The Uncertainty Relation in Linear and Nonlinear Quantum Mechanics and Quantum Fluctuation Effect

a) The uncertainty relation in linear quantum mechanics

The uncertainty relation in linear quantum mechanics is an important representation and also a problem that has troubled many scientists. Whether this is an intrinsic feature of microscopic particles or a result brought by the linear quantum mechanics or by the measuring instruments, it results in a long-term controversy in physics. How do we understand this correctly? Obviously, it is closely related to the wave-corpuscle duality of microscopic particles. Since we have established nonlinear quantum mechanics which differs from linear quantum mechanics, we could expect that the uncertainty relation in the nonlinear quantum mechanics is different from that in linear quantum mechanics [26-27]. The significance of the uncertainty relation can be revealed by comparing the linear and nonlinear quantum theories.

It is well known that the uncertainty relation in the linear quantum mechanics can be obtained from [1-6]

$$I(x)=\int\left|\left(x\Delta\hat{A}+i\Delta\hat{B}\right)y\ (\vec{r},t)\right|^2 d\vec{r}\geq 0 \quad (80)$$

or

$$\bar{\hat{F}}(x)=\int y^*(\vec{r},t)\hat{F}\left[\hat{A}(\vec{r},t),\hat{B}(\vec{r},t)\right]y\ (\vec{r},t)\,d\vec{r}$$

In the coordinate representation, $\hat{A}$ and $\hat{B}$ are operators of two physical quantities, for example, position and momentum, or energy and time, and satisfy the commutation relation $\left[\hat{A},\hat{B}\right]=i\hat{C}$, $y\ (x,t)$ and $y^*(x,t)$ are wave functions of the microscopic particle satisfying the linear Schrodinger equation and its conjugate equation, respectively, $\hat{F}=\left(\Delta Ax+\Delta B\right)^2$, ($\Delta\hat{A}=\hat{A}-\bar{A}$, $\Delta\hat{B}=\hat{B}-\bar{B}$, $\bar{A}$ and $\bar{B}$ are the average values of the physical

quantities in the state denoted by $y\ (x,t)$), is an operator of physical quantity related to $\bar{A}$ and $\bar{B}$, x is a real parameter.

After some simplifications, we can get

$$I=\bar{F}=\overline{\Delta\hat{A}^2}x^2+2\overline{\Delta\hat{A}\Delta\hat{B}}x+\overline{\Delta\hat{B}^2}\geq 0$$

or

$$\overline{\Delta\hat{A}^2}x^2+\bar{\hat{C}}x+\overline{\Delta\hat{B}^2}\geq 0 \tag{81}$$

Using mathematical identities, this can be written as

$$\overline{\Delta\hat{A}^2}\,\overline{\Delta\hat{B}^2}\geq\frac{\bar{\hat{C}}^2}{4} \tag{82}$$

This is the uncertainty relation in linear quantum mechanics. From the above derivation we see that the uncertainty relation was obtained based on the fundamental hypotheses of linear quantum mechanics, including properties of operators of the mechanical quantities, the state of particle represented by the wave function, which satisfies the linear Schrödinger equation (1), the concept of average values of mechanical quantities and the commutation relations and eigenequation of operators. Therefore, we can conclude that the uncertainty relation Eq. (82) is a necessary result of the linear quantum mechanics. Since the linear quantum mechanics only describes the wave nature of microscopic particles, the uncertainty relation is a result of the wave feature of microscopic particles, and it inherits the wave nature of microscopic particles. This is why its coordinate and momentum cannot be determined simultaneously. This is an essential interpretation for the uncertainty relation Eq. (82) in linear quantum mechanics. It is not related to measurement, but closely related to the linear quantum mechanics. In other words, if linear quantum mechanics could correctly describe the states of microscopic particles, then the uncertainty relation should also reflect the peculiarities of microscopic particles.

Equation (81) can be written in the following form:

$$\hat{F}=\overline{\Delta\hat{A}^2}\left(x+\frac{\overline{\Delta\hat{A}\Delta\hat{B}}}{\overline{\Delta\hat{A}^2}}\right)^2+\overline{\Delta\hat{B}^2}-\frac{\left(\overline{\Delta\hat{A}\Delta\hat{B}}\right)^2}{\overline{\Delta\hat{A}^2}}\geq 0$$

or

$$\overline{\Delta\hat{A}^2}\left(x+\frac{\bar{\hat{C}}}{4\overline{\Delta\hat{A}^2}}\right)^2+\overline{\Delta\hat{B}^2}-\frac{\left(\bar{\hat{C}}\right)^2}{4\overline{\Delta\hat{A}^2}}\geq 0 \tag{83}$$

This shows that $\overline{\Delta\hat{A}^2} \neq 0$, if $\left(\overline{\Delta\hat{A}\Delta\hat{B}}\right)^2$ or $\bar{\hat{C}}^2/4$ is not zero, else, we cannot obtain Eq.(82) and $\overline{\Delta\hat{A}^2}\,\overline{\Delta\hat{B}^2} > \overline{\left(\Delta A\Delta B\right)^2}$ because when $\overline{\Delta\hat{A}^2} = 0$, Eq. (83) does not hold. Therefore, $\left(\overline{\Delta\hat{A}^2}\right) \neq 0$ is a necessary condition for the uncertainty relation Eq. (82), $\overline{\Delta\hat{A}^2}$ can approach zero, but cannot be equal to zero. Therefore, in linear quantum mechanics, the right uncertainty relation should take the form [26-27]:

$$\overline{\Delta\hat{A}^2\Delta\hat{B}^2} > \frac{\left(\bar{\hat{C}}^2\right)^2}{4} \tag{84}$$

b) The uncertainty relation in nonlinear quantum mechanics

We now return to the uncertainty relation in nonlinear quantum mechanics. Since microscopic particles in NLQM is a soliton and they have wave-corpuscle duality, and there is no fundamental hypothesis in the nonlinear quantum mechanics, derivation of the uncertainty relation should be different from that in the linear quantum theory given above. We can also expect that the uncertainty relation in nonlinear quantum mechanics [26-27] is different from Eq.(84).

We now derive this relation for position and momentum of a microscopic particle depicted by the nonlinear Schrodinger Equation (5) with V(x,t)= $A(f)=0$, with a solution , f_s, as given in Eq.(9), which is now represented by

$$f_s(x',t') = 2h\sqrt{2/b}\,\mathrm{sec}\,h[2h(x'-x_o^{'}) - 8hxt']\exp[-4i(x^2-h^2)t' - 2ixx' + iq] \tag{85}$$

The function $f_s\left(x^{'},t^{'}\right)$ is a square integrable function localized at $x_0^{'}=0$ in the position space. If the microscopic particle is localized at $x_0^{'} \neq 0$, it satisfies the nonlinear Schrodinger equation,

$$if_{t'} + \frac{1}{2}f_{x'x'} + |f|^2 f = 0 \tag{86}$$

for b=1. The Fourier transform of this function [26-27] is

$$f_s\left(p,t^{'}\right) = \frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty} f_s\left(x^{'},t^{'}\right)e^{-ipx^{'}} \tag{87}$$

It shows that $f_s\left(p,t^{'}\right)$ is localized at p in momentum space. For Eq.(87), the Fourier transform is explicitly given by

$$f_s\left(p,t^{'}\right)=\sqrt{\frac{p}{2}}\,\mathrm{sec\,h}\left[\frac{p}{2h}\left(p-2\sqrt{2}x\right)\right]e^{4i(h^2+x^2-px/2\sqrt{2})t^{'}-i\left(p-2\sqrt{2}x\right)x_0^{'}} \tag{88}$$

The results in Eqs. (87) and (88) show that the microscopic particle is localized not only in position space in the shape of soliton, but also in the momentum space in a soliton. For convenience, we introduce the normalization coefficient A_0 in Eqs. (85) and (88), then obviously $A_o^2=\frac{1}{4\sqrt{2}}h$,the position of the mass centre of the microscopic particle, $\left\langle x^{'}\right\rangle$, and its square, $\left\langle x^{'2}\right\rangle, at\ \ t^{'}=0$ are given by

$$\left\langle x^{'}\right\rangle=\int_{-\infty}^{\infty}dx^{'}\left|f_s\left(x^{'}\right)\right|^2,\quad \left\langle x^{'2}\right\rangle=\int_{-\infty}^{\infty}dx^{'}x^{'2}\left|f_s\left(x^{'}\right)\right|^2.$$

We can thus find that

$$\left\langle x^{'}\right\rangle=4\sqrt{2}hA_0^2x_0^{'},\quad \left\langle x^{'2}\right\rangle=\frac{A_0^2p^2}{12\sqrt{2}h}+4\sqrt{2}A_0^2hx_0^{'2}$$

respectively. Similarly, the momentum of the mass center of the microscopic particle, $\left\langle p\right\rangle$, and its square , $\left\langle p^2\right\rangle$, are given by

$$\left\langle p\right\rangle=\int_{-\infty}^{\infty}p\left|\hat{f}_s\left(p\right)\right|^2dp,\quad \left\langle p^2\right\rangle=\int_{-\infty}^{\infty}p^2\left|\hat{f}_s\left(p\right)\right|^2dp$$

which yield

$$\left\langle p\right\rangle=16A_0^2hx,\quad \left\langle p^2\right\rangle=\frac{32\sqrt{2}}{3}A_0^2h^3+32\sqrt{2}A_0^2hx^3 \tag{89}$$

The standard deviations of the position $\Delta x^{'}=\sqrt{\left\langle x^{'2}\right\rangle-\left\langle x^{'}\right\rangle^2}$ and the momentum $\Delta p=\sqrt{\left\langle p^2\right\rangle-\left\langle p\right\rangle^2}$ are given by respectively.

$$\begin{aligned}(\Delta x')^2 &= A_0^2\left[\frac{p^2}{12h}+4h x_0'^2\left(1-4\sqrt{2}h A_0^2\right)\right]=\frac{p^2}{96h^2},\\ (\Delta p)^2 &= 32\sqrt{2}A_0^2\left[\frac{1}{3}h^3+hx^2\left(1-4\sqrt{2}h A_0^2\right)\right]=\frac{8}{3}h^2,\end{aligned} \tag{90}$$

Thus we obtain the uncertainty relation between position and momentum for the microscopic particle depicted by nonlinear quantum mechanics, Eq.(86)

$$\Delta x'\Delta p=\frac{p}{6} \tag{91}$$

This result is not related to the features of the microscopic particle (soliton) depicted by the nonlinear Schrodinger equation in nonlinear quantum mechanics because Eq. (91) has nothing to do with characteristic parameters of the nonlinear Schrodinger equation. p in Eq. (91) comes from of the integral coefficient $1/\sqrt{2p}$. For a quantized microscopic particle, p in Eq. (91) should be replaced by $p\,\mathrm{h}$, because Eq. (87) is replaced by

$$f_s(p,t')=\frac{1}{\sqrt{2p\,\mathrm{h}}}\int_{-\infty}^{\infty}dx' f_s(x',t')e^{-ipx'/\mathrm{h}}.$$

The corresponding uncertainty relation of the quantum microscopic particle in nonlinear quantum mechanics is given by

$$\Delta x\Delta p=\frac{p\,\mathrm{h}}{6}=\frac{h}{12} \tag{92}$$

The uncertainty relation in Eq. (92) or Eq. (91) [26-27] is different from that in linear quantum mechanics Eq. (84), i.e., $\Delta x\Delta p>h/2$. However, the minimum value $\Delta x\Delta p=h/2$ has not been observed in practical systems in linear quantum mechanics up to now except for the coherent and squeezed states of microscopic particles. The relation Eq.(91) cannot be obtained from the solutions of linear Schrodinger equation. Practically, we can only get $\Delta x\Delta p>h/2$ from Eq.(84), but not $\Delta x\Delta p=h/2$, in linear quantum mechanics.

c) The uncertainty relations of coherent states

As a matter of fact, we can represent one-quantum coherent state of harmonic oscillator by [26-27]

$$|a\rangle=\exp\left(a\hat{b}^{+}-a^{*}\hat{b}\right)|0\rangle=e^{-a^2/2}\sum_{n=0}^{\infty}\frac{a^n}{\sqrt{n-1}}\hat{b}^{+n}|0\rangle,$$

in the number picture, which is a coherent superposition of a large number of microscopic particles (quanta). Thus

$$\langle a|\hat{x}|a\rangle=\sqrt{\frac{\mathrm{h}}{2wm}}\left(a+a^{*}\right),\quad \langle a|\hat{p}|a\rangle=i\sqrt{\mathrm{h}mw/2}\left(a-a^{*}\right),$$

and

$$\langle a|\hat{x}^{2}|a\rangle=\frac{\mathrm{h}}{2wm}\left(a^{*2}+a^{2}+2aa^{*}+1\right),\quad \langle a|\hat{p}^{2}|a\rangle=\frac{\mathrm{h}wm}{2}\left(a^{*2}+a^{2}-2aa^{*}-1\right),$$

where

$$\hat{x}=\sqrt{\frac{\mathrm{h}}{2wm}}\left(\hat{b}+\hat{b}^{+}\right),\quad \hat{p}=i\sqrt{\frac{\mathrm{h}wm}{2}}\left(\hat{b}^{+}-\hat{b}\right),$$

and $\hat{b}^{+}\left(\hat{b}\right)$ is the creation (annihilation) operator of the microscopic particle (quantum), a and a^{*} are some unknown functions, w is the frequency of the particle, m is its mass. Thus we can get

$$(\Delta x)^{2}=\frac{\mathrm{h}}{2wm},\quad (\Delta p)^{2}=\frac{\mathrm{h}wm}{2},\quad \langle\Delta x\rangle^{2}\langle\Delta p\rangle^{2}=\frac{h^{2}}{4} \tag{93}$$

$$\frac{\Delta x}{\Delta p}=\frac{1}{wm}\text{ , or }\Delta p=(wm)\Delta x$$

For the squeezed state of the microscopic particle: $|b\rangle=\exp\left[b\left(b^{+2}-b^{2}\right)\right]|0\rangle$, which is a two-microscopic particle (quanta) coherent state, we can find that

$$\langle b|\Delta x^{2}|b\rangle=\frac{\mathrm{h}}{2mw}e^{4b},\langle b|\Delta p^{2}|b\rangle=\frac{\mathrm{h}mw}{2}e^{-4b},$$

using a similar approach as the above . Here b is the squeezed coefficient and $|b|<1$. Thus,

$$\Delta x\Delta p=\frac{h}{2},\quad \frac{\Delta x}{\Delta p}=\frac{1}{mw}e^{8b},\text{ or }\Delta p=\Delta x(wm)e^{-8b} \tag{94}$$

This shows that the momentum of the microscopic particle (quantum) is squeezed in the two-quanta coherent state compared to that in the one-quantum coherent state.

From the above results, we see that both one-quantum and two-quanta coherent states satisfy the minimal uncertainty principle. This is the same as that of the nonlinear quantum states in nonlinear quantum mechanics. We can conclude that a coherent state is a kind of nonlinear quantum state, and the coherence of quanta is a nonlinear phenomenon, instead of a linear effect.

As is known, the coherent state satisfies the classical equation of motion, in which the fluctuation in the number of particles approaches zero, which is a classically steady wave. In fact, according to quantum theory, the coherent state of a harmonic oscillator at time t can be represented by

$$|a,t\rangle = e^{-i\hat{H}t}|a\rangle = e^{-i\hbar w\left(\hat{b}^{+}\hat{b}+1/2\right)t}|a\rangle = e^{-i\hbar wt/2-|a|^2/2}\sum_{n=0}^{\infty}\frac{a^n e^{-i\hbar nwt}}{\sqrt{n!}}|n\rangle$$

$$= e^{-i\hbar wt/2}\left|a\,e^{-i\hbar wt}\right\rangle, \quad \left(|n\rangle = \left(b^{+}\right)^n|0\rangle\right)$$

This shows that the shape of a coherent state can be retained during its motion. This is the same as that of a microscopic particle (soliton) in nonlinear quantum mechanics. The mean position of the particle in the time-dependent coherent state is

$$\langle a,t|x|a,t\rangle = \langle a|e^{iHt/\hbar}xe^{-iHt/\hbar}|a\rangle = \left\langle a\left|x-\frac{it}{\hbar}[x,H]+\frac{(-it)^2}{2!\hbar^2}[[x,H],H]+\mathrm{L}\right|a\right\rangle$$

$$= \left\langle a\left|x+\frac{pt}{m}-\frac{1}{2!}t^2w^2x+\mathrm{L}\right|a\right\rangle = \left\langle a\left|x\cos wt+\frac{p}{mw}\sin wt\right|a\right\rangle = \sqrt{\frac{2\hbar}{mw}}|a|\cos\left(wt+q\right) \tag{95}$$

where $q=\tan^{-1}\left(\frac{y}{x}\right)$, $x+iy=a$, $[x,H]=\frac{i\hbar p}{m}$, $\quad [p,H]=-i\hbar mw^2x.$

Comparing (95) with the solution of a classical harmonic oscillator

$$x=\sqrt{\frac{2E}{mw^2}}\cos\left(wt+q\right), \quad E=\frac{p^2}{2m}+\frac{1}{2}mw^2x^2$$

we find that they are similar, with

$$E=\hbar wa^2=\langle a|H|a\rangle-\langle 0|H|0\rangle, \quad H=\hbar w\left(b^{+}b+\frac{1}{2}\right).$$

Thus, we can say that the center of the coherent state-packet indeed obeys the classical law of motion, which is the same as the law of motion of microscopic particles in nonlinear quantum mechanics discussed in Eqs. (70)-(71).

We can similarly obtain

$$\langle a,t|p|a,t\rangle=-\sqrt{2m\hbar w}\,|a|\sin(wt+q),\langle a,t|x^2|a,t\rangle=\frac{2\hbar}{wm}\left[|a|^2\cos^2(wt+q)+\frac{1}{4}\right],$$

$$\langle a,t|p^2|a,t\rangle=2m\hbar w\left[|a|^2\sin^2(wt+q)+\frac{1}{4}\right]$$

and

$$[\Delta x(t)]^2=\frac{\hbar}{2wm},\quad [\Delta p(t)]^2=\frac{1}{2}mw\hbar,\Delta x(t)\Delta p(t)=\frac{\hbar}{2} \tag{96}$$

This is the same as Eq. (92). It shows that the minimal uncertainty principle for the coherent state is retained at all times, *i.e.*, the uncertainty relation does not change with time t.

The mean number of quanta in the coherent state is given by

$$\bar{n}=\langle a|\hat{N}|a\rangle=\langle a|\hat{b}^+\hat{b}|a\rangle=a^2,\quad \langle a|\hat{N}^2|a\rangle=|a|^4+|a|^2$$

Therefore, the fluctuation of the quantum in the coherent state is

$$\Delta n=\sqrt{\langle a|\hat{N}^2|a\rangle-\left(\langle a|\hat{N}^2|a\rangle\right)^2}=|a|.$$

which leads to

$$\frac{\Delta n}{n}=\frac{1}{|a|}=1.$$

It is thus obvious that the fluctuation of the quantum in the coherent state is very small. The coherent state is quite close to the feature of soliton and solitary wave.

These properties of coherent states are also similar to those of microscopic particles described by the nonlinear Schrodinger equation , the f^4-equation , or the Sine-Gordon equation in nonlinear quantum mechanics. In practice, the state of a microscopic particle in nonlinear quantum mechanics can always be represented by a coherent state, for example, the Davydov's wave functions, both $\mathrm{ID}_1>$ and $\mathrm{ID}_2>$, [46] and Pang's wavefunction [47-48] of exciton-solitons in protein molecules and the wavefunction in acetanilide [49-50]; the wavefunction of proton transfer in hydrogen-bonded systems [51-53] and the BCS's wave function in superconductors[54], etc. Hence, the coherence of particles is a kind of nonlinear phenomenon that occurs only in nonlinear quantum mechanics. It does not belong to systems described by linear quantum mechanics, because the coherent state cannot be obtained by superposition of linear waves, such as plane wave, de Broglie wave, or Bloch wave, which

are solutions of the linear Schrodinger equation in linear quantum mechanics. Therefore, the minimal uncertainty relation Eq. (92), as well as Eqs. (94) and (96), are only applicable to microscopic particles in nonlinear quantum mechanics. In other words, only microscopic particles in nonlinear quantum mechanics satisfy the minimal uncertainty principle. It reflects the wave-corpuscle duality of microscopic particles because it holds only if the duality exists.

This uncertainty principle also suggests that the position and momentum of the microscopic particle can be simultaneously determined in a certain degree and range. A rough estimate for the size of the uncertainty can be given. If it is required that $f_s(x,t)$ in Eq.(85) or $f_s(p,t)$ in Eq. (87) satisfies the admissibility condition *i.e.*, $f_s(0)\approx 0$, we choose $x=140,\ h=\sqrt{300/0.253}/2\sqrt{2}$ and $\overset{1}{x}_0=0$ in Eq.(85) (In fact, in such a case we can get $f_s(0)\approx 10^{-6}$, thus the admissibility condition can be satisfied). We then get $\Delta x\approx 0.02624$ and $\Delta p\approx 19.893,$ according to (91) and (92). These results show that the position and momentum of microscopic particles in nonlinear quantum mechanics can be simultaneously determined within a certain approximation.

Pang *et al.* [55-56] also calculated the uncertainty relation and quantum fluctuations and studied their properties in nonlinearly coupled electron-phonon systems based on the Holstein model by a new ansatz including the correlations among one-phonon coherent and two-phonon squeezing states and polaron state proposed by himself. Many interesting results were obtained. The minimum uncertainty relation takes different forms in different systems which are related to the properties of the microscopic particles. Nevertheless, the minimum uncertainty relation in Eq. (92) holds for both the one-quantum coherent state and two-quanta squeezed state. These works enhanced our understanding of the significance and nature of the minimum uncertainty relation.

(d) Quantum fluctuation of particles described by quantum nonlinear Schrodinger equation

Finally, we determine the uncertainty relation of the microscopic particle described by quantum nonlinear Schrodinger equation, arising from the quantum fluctuation effect in the nonlinear quantum field theory. The quantum theory was discussed by Lai and Haus et. al[57, 26-27] based on the nonlinear Schrodinger equation. They think that a superposition of a subclass of bound state $|n,P\rangle$, characterized by number of the boson, for example, photon or phonon, and the momentum of the center of the mass P, can reproduce the expectation values of the microscopic particle (soliton) in the limit where the average number of the bosons (phonons) are larger; Lai et.al. refer to these states formed by the superposition of $|n,P\rangle$ as a fundamental soliton states. We here discuss the quantum fluctuation of MIPs(solitons) depicted by NLSE (5) at V(x,t)= $A(f)=0$ by means of Lai et al's method[57]. In nonlinear quantum theory, the quantized dynamic equation in the second quantized picture is given by

$$i\mathrm{h}\frac{\partial}{\partial t}\hat{f}(x,t)=-\frac{\mathrm{h}^2}{2m}\frac{\partial^2}{\partial x^2}\hat{f}(x,t)+2b\hat{f}^{+}(x,t)\hat{f}(x,t)\hat{f}(x,t) \qquad (97)$$

The operators $\hat{\psi}(x,t)$ and $\hat{\psi}^{+}(x,t)$ are the annihilation and creation operators of field of a quantum at a "point" *x* and "time" *t*, they satisfy the commutation relation:

$$[\hat{\psi}(x'',t),\hat{\psi}^{+}(x,t)]=d(x-x''),[\hat{\psi}(x'',t),\hat{\psi}(x,t)]=[\hat{\psi}^{+}(x'',t),\hat{\psi}^{+}(x,t)]=0 \qquad (98)$$

The corresponding quantum Hamiltonian is given by

$$\hat{H}=\frac{\hbar^2}{2m}\int \hat{\psi}_x^{+}(x,t)\hat{\psi}_x(x,t)dx+b\int \hat{\psi}^{+}(x,t)\hat{\psi}^{+}(x,t)\hat{\psi}(x,t)\hat{\psi}(x,t)dx \qquad (99)$$

In the Schrodinger picture, the time evolution of the system is described by

$$i\hbar\frac{d}{dt}|\Phi\rangle=\hat{H}_s|\Phi\rangle \qquad (100)$$

with the commutation relation:

$$[\hat{\psi}(x''),\hat{\psi}^{+}(x)]=d(x-x''),[\hat{\psi}(x''),\hat{\psi}(x)]=[\hat{\psi}^{+}(x''),\hat{\psi}^{+}(x)]=0 \qquad (101)$$

where $\hat{\psi}(x)$ and $\hat{\psi}^{+}(x)$ are the field operators in the Schrodinger representation. The corresponding quantum Hamiltonian is given by

$$\hat{H}_s=\frac{\hbar^2}{2m}\int \hat{\psi}_x^{+}(x)\hat{\psi}_x(x)dx+b\int \hat{\psi}^{+}(x)\hat{\psi}^{+}(x)\hat{\psi}(x)\hat{\psi}(x)dx \qquad (102)$$

The many-particle state $|\Phi\rangle$ can be built up from the $n-$particle states given by

$$|\Phi\rangle=\sum_n a_n\int\frac{1}{\sqrt{n!}}f_n(x_1...,x_n,t)\hat{\psi}^{+}(x_1)...\hat{\psi}^{+}(x_n)dx_1...dx_n|0\rangle \qquad (103).$$

The quantum theory based on Eq.(103) describes an ensemble of bosons interacting via a $d-$potential. Note that $\hat{H}$ preserves both the particle number.

$$\hat{N}=\int\hat{\psi}^{+}(x)\hat{\psi}(x)dx \quad (104)$$

and the total momentum

$$\hat{P}=i\frac{\mathrm{h}}{2}\int\left[\frac{\partial}{\partial x}\hat{f}^{+}(x)\hat{f}(x)-\hat{f}^{+}(x)\frac{\partial}{\partial x}\hat{f}(x)\right]dx \tag{105}$$

Lai *et al.*[55] proved that the boson number and momentum operator commute, so that common eigenstates of $\hat{H}$, $\hat{P}$ and $\hat{N}$ exist in such a case. In the case of a negative ratio b, the interaction between the bosons is attractive and Hamiltonian Eq. (97) has bound states. A subset of these bound states is characterized solely by the eigenvalues of $\hat{N}$ and $\hat{P}$:

$$f_{n,p}=N_{n,}\exp\left(ip\sum_{j=1}^{\infty}x_j+\frac{b}{2}\sum_{1\le i,j<n}^{\infty}\left|x_i-x_j\right|\right), \tag{106}$$

where $N_n=\sqrt{\dfrac{(n-1)!|b|^{n-1}}{2p}}$.

Thus

$$f_n(x_1,\mathrm{L}\ ,x_n,t)=\int dpg_n(p)\,f_np(x_1,\mathrm{L}\ ,x_n,t)e^{-iE(n,p)t}, \tag{107}$$

where $g_n(p)=\sqrt{g(p)}e^{-inpx_0}$, and $g(p)=\dfrac{exp\left\{-(p-p_0)^2\Big/\left|2(\Delta p)^2\right|\right\}}{\sqrt{2p(\Delta p)^2}}$

Using $f_{n,p}$ given in Eq. (106), we find that $|n,P\rangle$ decays exponentially with separation between an arbitrary pair of bosons. It describes an $n-$particle soliton moving with momentum $P=\mathrm{h}np$ and energy $E(n,p)=np^2-|b|^2(n^2-1)n/12$. By construction, the quantum number p in this wave function is related to the momentum of the mass centre of the n interacting bosons, which is now defined as

$$\hat{X}=\lim_{e\to 0}\int x\hat{f}^{+}(x)\hat{f}(x)dx\left(e+\hat{N}\right)^{-1} \tag{108}$$

with $\left[\hat{X},\hat{P}\right]=i\mathrm{h}$

The limit of $e\to 0$ is introduced to regularize the position operator for the vacuum state.

We are interested in the fluctuations of Eqs. (104), (105) and (106) for a state $|\Phi(t)\rangle$ with a large average Boson number and a well-defined mean field. Kartner and Boiven[56] decomposed the field operator in its mean value and a remainder which is responsible for the quantum fluctuations.

$$\hat{f}(x)=\left\langle y'(0)\left|\hat{f}^{+}(x)\right|y(0)\right\rangle+\hat{f}_1(x),\left[\hat{f}_1(x),\hat{f}_1^{+}(x')\right]=d(x-x'),\left[\hat{f}_1(x),\hat{f}_1(x')\right]=0 \tag{109}$$

Since the field operator $\hat{f}$ is time independent in the Schrodinger representation, we can then choose $t=0$ for definiteness. Inserting Eq.(109) into Eqs.(104), (105) and (108) and neglecting terms of second and higher order in the noise operator, Kartner et al. obtained that

$$\hat{N}=n_0+\Delta\hat{n},n_0=\int dx\left(\left\langle\hat{f}^{+}(x)\right\rangle\left\langle\hat{f}(x)\right\rangle\right),\Delta\hat{n}=\int dx\left(\left\langle\hat{f}^{+}(x)\right\rangle\hat{f}_1(x)\right)+c.c.,$$

$$\hat{P}=\mathrm{h}n_0p_0+\mathrm{h}n_0\Delta\hat{p},p_0=\frac{i}{n_0}\int dx\left\langle\hat{f}_x^{+}(x)\right\rangle\left\langle\hat{f}(x)\right\rangle,\Delta\hat{p}=\frac{i}{n_0}\int dx\left\langle\hat{f}_x^{+}(x)\right\rangle\hat{f}_1(x)+c.c.,$$

$$\hat{X}=x_0\left(1-\frac{\Delta\hat{n}}{n_0}\right)+\Delta\hat{x},x_0=\frac{1}{n_0}\int dxx\left\langle\hat{f}^{+}(x)\right\rangle\left\langle\hat{f}(x)\right\rangle,\Delta\hat{x}=\frac{1}{n_0}\int dxx\left\langle\hat{f}^{+}(x)\right\rangle\hat{f}(x)+c.c.$$

where $\Delta\hat{x}$ is the deviation from the mean value of the position operator, $\Delta\hat{n},\Delta\hat{p},$and $\Delta\hat{x}$ are linear in the noise operator. Because the third- and fourth-order correlators of $\hat{f}_1$ and $\hat{f}_1^{+}$ are very small, they can be neglected in the limit of large n_0. Note that $\Delta\hat{n},\Delta\hat{p},$and $\Delta\hat{x}$ are all quadratures of the noise operator with $\Delta\hat{p}$ and $\Delta\hat{x}$ being conjugate variables. To complete this set, they introduce a quadrature variable conjugate to $\Delta\hat{n}$,

$$\Delta\hat{q}=\frac{1}{n_0}\int dx\left\{i\left[\hat{f}^{+}(x)+x\left\langle\hat{f}_x^{+}(x)\right\rangle\right]-p_0x\left\langle\hat{f}_x^{+}(x)\right\rangle\right\}\hat{f}_1(x)+c.c.$$

As is known, if the propagation distance is not too large, the mean value of the field is given to the first order by the classical soliton solution

$$\left\langle\hat{f}(x)\right\rangle=f_{0,n_0}(x,t)\left[1+O\left(\frac{1}{n_0}\right)\right]$$

with

$$f_{0,n_0}(x,t)=\frac{n_0\sqrt{|b|}}{2}\exp\left[i\Omega_{nl}-ip_0^2t+ip_0(x-x_0)+iq_0\right]\times\mathrm{sech}\left[\frac{n_0|b|}{2}(x-x_0-2p_0t)\right], \tag{110}$$

and the nonlinear phase shift $\Omega_{nl} = n_0^2 |b|^2 t/4$. If $p_0 = x_0 = q_0 = 0$, Kartner et al obtained the following for the fluctuation operators in the Heisenberg picture,

$$\Delta\hat{n}(t) = \int dx\left[f_{-n}(x)^* F'_{nl} + c.c \right], \Delta\hat{q}(t) = \int dx\left[f_{-q}(x)^* F'_{nl} + c.c \right],$$

$$\Delta\hat{p}(t) = \int dx\left[f_{-p}(x)^* F'_{nl} + c.c \right], \Delta\hat{x}(t) = \int dx\left[f_{-x}(x)^* F'_{nl} + c.c \right],$$

with $F'_{nl} = e^{i\Omega_{nl}} \hat{f}_1(x,t)$,
and the set of adjoint functions

$$f_{-n}(x) = \frac{n_0\sqrt{|b|}}{2}\sec h(x_{n_0}),\ f_{-q}(x) = \frac{i\sqrt{|b|}}{2}\left[\sec h(x_{n_0}) + x_{n_0}\frac{d}{dx_{n_0}}\sec h(x_{n_0})\right],$$

$$f_{-p}(x) = -\frac{in_0\sqrt{|b|^3}}{4}\frac{d}{dx_{n_0}}\sec h(x_{n_0}),\ f_{-x}(x) = \frac{1}{n_0\sqrt{|b|}}x_{n_0}\sec h(x_{n_0}),$$

where $x_{n_0} = \frac{1}{2}n_0|b|x$

For a coherent state defined by

$$\hat{f}(x)\left|\Phi_{0,n_0}\right\rangle = f_{0,n_0}(x)\left|\Phi_{0,n_0}\right\rangle,\ \ \hat{f}_1(x)\left|f_{0,n_0}\right\rangle = 0$$

where

$$\left|\Phi_{0,n_0}\right\rangle = \exp\left\{\int dx\left[f_{0,n_0}(x)\hat{f}^+(x) - f^*{}_{0,n_0}(x)\hat{f}^+(x)\right]\right\}|0\rangle$$

f_{0,n_0} has been given Eq. (110). Kartner *et. al.* further obtained that

$$\left\langle\Delta\hat{n}_0^2\right\rangle = n_0,\ \left\langle\Delta\hat{q}_0^2\right\rangle = \frac{0.6075}{n_0},\ \left\langle\Delta\hat{p}_0^2\right\rangle = \frac{1}{3n_0 t_0^{\,2}},\ \ \left\langle\Delta\hat{x}_0^2\right\rangle = \frac{1.645 t_0^{\,2}}{2n_0},$$

where $t_0^{\,2} = 2/n_0|b|$ is the width of the microscopic particle (soliton). The uncertainty products of Boson number and phase, momentum and position are, respectively,

$$\left\langle \Delta\hat{n}_0^2 \right\rangle \left\langle \Delta\hat{q}_0^2 \right\rangle = 0.6075 \geq 0.25, n_0^2 \left\langle \Delta\hat{p}_0^2 \right\rangle \left\langle \Delta\hat{x}_0^2 \right\rangle = 0.27 \geq 0.25,$$

Here the quantum fluctuation of the coherent state is white, i.e.,

$$\left\langle \hat{f}_1(x)\hat{f}_1(y) \right\rangle = \left\langle \hat{f}_1^+(x)\hat{f}_1(y) \right\rangle = 0$$

However, the quantum fluctuation of the soliton cannot be written because the particle interaction introduces correlations between them. Thus, Kärner and Boivin[58] assumed a fundamental soliton state with a Poissonian distribution for the boson number $p_n = \frac{n_0^n}{n!} e^{-n_0}$ and a Gaussian distribution for the momentum Eq.(73) with a width $\left\langle \Delta p_0^2 \right\rangle = n_0 |b|^2 / 4m$, where m is a parameter of the order of unity compared to n_0. They finally obtained the minimum uncertainty values:

$$\left\langle \Delta\hat{q}_0^2 \right\rangle = \frac{0.25}{\left\langle \Delta\hat{n}^2 \right\rangle} = \frac{0.25}{n_0}\left[1 + O\left(\frac{1}{n_0}\right)\right], \text{and } \left\langle \Delta\hat{x}_0^2 \right\rangle = \frac{0.25}{\left\langle n_0^2 \right\rangle} = \frac{0.25 m t_0^2}{n_0}\left[1 + O\left(\frac{1}{n_0}\right)\right]$$

up to order $1/n_0$ for the corresponding initial fluctuations in MIP (soliton) phase and timing. Thus, at t=0 the fundamental soliton with the given Boson number and momentum distributions is a minimum uncertainty state in the four collective variables, the Boson number, phase momentum and position, up to the terms of O($1/n_0$), which are of the form [57,26-27]

$$\left\langle \Delta\hat{n}_0^2 \right\rangle \left\langle \Delta\hat{q}_0^2 \right\rangle = 0.25\left[1 + O\left(\frac{1}{n_0}\right)\right], \text{ and } n_0^2 \left\langle \Delta p_0^2 \right\rangle \left\langle \Delta\hat{x}_0^2 \right\rangle = 0.25\left[1 + O\left(\frac{1}{n_0}\right)\right] \quad (111)$$

These are the uncertainty relations arising from the quantum fluctuations in nonlinear quantum mechanics of MIP described by NLSE. They are the same as Eqs.(92)-(94). Therefore, we can conclude that the uncertainty relation in NLQM takes the minimum values regardless whether a state is coherent or squeezed, a system is classical or quantum.

In light of the above discussion, we can distinguish the motions of particles in the linear quantum mechanics, nonlinear quantum mechanics, and classical mechanics by means of the uncertainty relation. When the motion of the particles satisfies $\Delta x \Delta p > h/2$ *or* $p/6$, the particles obey laws of motion in linear quantum mechanics, and the particles are some waves. When the motion of the particles satisfies $\Delta x \Delta p = h/12$ *or* $p/6$, the particles obey laws of motion in nonlinear quantum mechanics, and the particles are solitons, exhibiting wave-corpuscle duality. If the motion of the particles satisfies $\Delta x \Delta p = 0$, then the particles can be treated as classical particles, with only corpuscle feature. The nonlinear quantum mechanics

introduced here makes physics more complete. Therefore, we can say that the nonlinear quantum mechanics is a new theory which is a new physical branch, it bridges the gap between the classical and linear quantum mechanics.

4. EIGENVALUE PROBLEM OF MICROSOPIC PARTICLES IN NLQM

4.1. The Eigenenergy Spectrum of the Hamiltonian of the Systems

In linear quantum mechanics (LQM), because the Hamiltonian of the systems is independent of the state wavefunction of the particle, the eigenenergy spectrum of the Hamiltonian operator of the systems can be easily obtained from its eigenequation, $\hat{H}\left|y\left(x,t\right)\right\rangle = E\left|y\left(x,t\right)\right\rangle$, where $\left|y\left(x,t\right)\right\rangle$ is its eigenwave-function in coordinate or particle number representation. It also is just a time-independent linear Schrodinger equation in the coordinate representation. However, in the NLQM we find that this method fails in the coordinate representation because the wavefunction of state of MIP is contained in the Hamiltonian operator of the systems; that is, the Hamiltonian operator depends on the state wave vector of the MIP, thus exact eigenvalues cannot be obtained. Therefore, we must use other methods to find the eigenenergies of the Hamiltonian operator of the MIPs. We have two ways to get the eigenenergies of the Hamiltonian operator. The first is that the energy of the MIP satisfying the NLSE can be obtained from $E=\int_{-\infty}^{\infty}\left\langle f(x,t)\right|\hat{H}\left|f(x,t)\right\rangle dx$ as used in Eq. (18), where H' = H is the Hamiltonian density of the systems which depends on the wave function $f\left(x,t\right)$. The second is that the Hamiltonian operator and state wavefunction of particles are all given in particle number representation, then we can find the eigenenergy spectrum of the Hamiltonian operator from its eigenequation, $\hat{H}'\left|f\left(x,t\right)\right\rangle = E\left|f\left(x,t\right)\right\rangle$, whether one mode motion or many mode motion. We often use the latter to find the eigenenergy of the Hamiltonian operator of the system.

We know that the wave function of a microscopic particle can be quantized by the creation and annihilation operators of the particle in the second quantum representation in NLQM. Then the Hamiltonian of a system described by the wave function $f\left(x,t\right)$ can be quantized by introducing creation and annihilation operators in the particle number representation or second quantization representation. Thus, we can calculate the eigenenergy spectrum by using the eigenequation of the quantum Hamiltonian and corresponding wavevector in number representation. This is basically how the eigenenergy spectra in the NLQM can be obtained. For convenience, we express the nonlinear Schrodinger Eq. (5) with A(f)=0 in the following discrete form:

$$i\hbar\frac{\partial f_j}{\partial t}=-\frac{\hbar}{2mr_0^2}(f_{j+1}-2f_j+f_{j-1})-b\left|f_j\right|^2 f_j+V(j,t)f,(j=1,2,3,...,J) \tag{112}$$

in lattice field, where r_0 is a spacing between two neighboring lattice points, j labels the discrete lattice points, J is total number of lattice points in the lattice field of the system. The vector form of the above equation in the lattice field is

$$[ih\frac{\partial}{\partial t}-\frac{h^2}{mr_0^2}-V(j,t)]\bar{f}=-eM\bar{f}-bdiag.(|f_1|^2,|f_2|^2...|f_a|^2)\bar{f}, \tag{113}$$

where $\bar{f}(x,t)$ is the column vector, $\bar{f}(x,t)$ =Col.(f_1 , f_2 ,... f_a), whose complex components, equation (113) is a vector NLSE with a modes of motion. In Eq. (113), b is a nonlinear parameter and a is a number of motion modes that exist in the systems. M=[M_{nl}]is an $a\times a$ real symmetric dispersion matrix, $e=h^2/2mr_0^2$. Here, n and l are integers denoting the modes of motion. The Hamiltonian and the particle number corresponding to Eq. (113), respectively, are

$$H=\sum_{N=1}^{a}\left(\eta w_0|f_n|^2-\frac{1}{2}b|f_n|^4\right)-e\sum_{n\neq l}^{a}M_{nl}f_nf_l, \text{ and } N=\sum_{N=L}^{a}|f_n|^2 \tag{114}$$

where $hw_0=h^2/2mr_0^2+V(j,t)$.

We have assumed that $V(j,t)$ are independent of j and t. In the canonical second quantization theory, the complex amplitude (f_n^* and f_n) become boson creation and annihilation operators ($\hat{B}_n^+$ and $\hat{B}_n$) in the number representation. If $|m_n>$ is an eigenfunction of a particular mode, then

$$\hat{B}_n^+|m_n\rangle=\sqrt{m_n+1}\,|m_n+1>,\hat{B}_n|m_n>=\sqrt{m_n}\,|m_n-1>\text{ and }\hat{B}_n|0>=0.$$

Since no particular ordering is specified in Eq.(114) we use the averages:

$$|f|^2\to\frac{1}{2}(\hat{B}_n^+\hat{B}_n+\hat{B}_n\hat{B}_n^+)$$

and

$$|f_n|^4\to\frac{1}{6}(\hat{B}_n^+\hat{B}_n^+\hat{B}_n\hat{B}_n+\hat{B}_n^+\hat{B}_n\hat{B}_n^+\hat{B}_n+\hat{B}_n^+\hat{B}_n\hat{B}_n\hat{B}_n^++\hat{B}_n\hat{B}_n^+\hat{B}_n\hat{B}_n^++\hat{B}_n\hat{B}_n\hat{B}_n^+\hat{B}_n^++\hat{B}_n\hat{B}_n^+\hat{B}_n^+\hat{B}_n)$$

with the Boson commutation rule $\hat{B}_n\hat{B}_n^+-\hat{B}_n^+\hat{B}_n=1$, Eq. (114) then becomes

$$\hat{H}=\sum_{n=1}^{a}[(\hbar w_0-\frac{1}{2}b)(\hat{B}_n^{+}\hat{B}_n+\frac{1}{2})-\frac{1}{2}b\hat{B}_n^{+}\hat{B}_n\hat{B}_n^{+}\hat{B}_n]-e\sum_{n\neq l}^{a}M_{nl}\hat{B}_n^{+}\hat{B}_l \qquad (115)$$

$$\hat{N}=\sum_{n=1}^{a}(\hat{B}_n^{+}\hat{B}_n^{+}+\frac{1}{2}) \qquad (116)$$

From now on, we will use the notation [m_1 , m_2 ,... m_a] to denote the products of number states| m_1 >| m_2 >…| m_a >. Thus, stationary states of the vector NLSE (114) must be eigenfunctions of both $\hat{N}$ and $\hat{H}$. Consider an m-quantum state (i.e., the mth excited level, m= $m_1+m_2+\ldots m_j$), with m< a . An eigenfunction of $\hat{N}$ can be established as

$$\left|f_m\right\rangle = C_1\ [\mathrm{m,0,0,\ldots,0}]+\ldots+\ C_2\ [\mathrm{0,m,0,0,\ldots,0}]+\ldots+\ C_i\ [\mathrm{0,0,0,\ldots,\ m\ ,}]+\ \ldots+\ C_{i+1}\ [\mathrm{m-1,m,0,0,\ldots,0}]+\ldots+C_p\,[\mathrm{0,0,0,\ldots,0,1,1\ldots,1}]. \qquad (117)$$

The number of terms in Eq.(117) is equal to the number of ways that m quanta can be placed on a sites, which is given by P=$\frac{(m+a-1)}{m!(a-1)!}$. The wave function | f_m > in Eq.(117) is an eigenfunction of $\hat{N}$ for any values of the $C_a^{'}$. Thus, we are free to choose these coefficients so that

$$\hat{H}|f_m>=\mathrm{E}|f_m>. \qquad (118)$$

Equation (118) requires that the column vector C=Col.(C_1 , C_2 ,… C_p) satisfies the matrix equation:

$$|\mathrm{H}-\mathrm{IE}|\mathrm{C}=0 \qquad (119)$$

where H is a $p\times p$ symmetric matrix with real elements. I is a $p\times p$ identity matrix, E is just the eigenenergy. Eq. (118) is an eigenvalue equation of quantum Hamiltonian operator (115) of the systems. We can find the eigenenergy spectra E_m of the systems from Eq. (119) for given parameters, e , w_0 , and b. Scott *et al.* [59-61] and Pang *et al.* [62-69] used this method to calculate the energy-spectra of vibrational excitations (quanta) in many nonlinear systems, for example, small molecules or organic molecular crystals and biomolecules. These results can be compared with the experimental data.

4.2. EIGENVALUE PROBLEM OF THE NONLINEAR SCHRODINGER EQUATION

In LQM we know that the time-independent linear Schrodinger equation is an eigenequation of the Hamiltonian operator in the coordinate representation. However, we do not know the meaning of the eigenvalue problem of the nonlinear Schrodinger equation, which is therefore a new problem. This problem comes from the Lax method. According to this method, for any nonlinear equation, $\frac{\partial}{\partial t} f(\mathrm{r}, t) =$ K($f(\mathrm{r}, t)$), where K($f(\mathrm{r}, t)$) is a nonlinear operator. If K($f(\mathrm{r}, t)$) is related to two linear operators $\hat{L}$ and $\hat{B}$, which depend on f and satisfy the Lax operator equation:

$$i\dot{\hat{L}} = \hat{B}\hat{L} - \hat{L}\hat{B} = [\hat{B}, \hat{L}], \tag{120}$$

where $t' = t/\eta$, then the eigenvalue l, which does not vary with time, and eigenfunction y of the nonlinear equation is determined by the eigenequation of operator $\hat{L}$ as follows

$$\hat{L}y = l\,y\ ;\ iy_{t'} = \hat{B}y \tag{121}$$

Thus, the eigenvector and eigenvalue of nonlinear systems are determined by the eigenvector and eigenvalue of the above linear operators. In general, concerning any types of nonlinear equation, its corresponding linear eigenequation and time-independent eigenvalue can always be found. For the NLSE in Eq. (5) with $V(\mathrm{r}, t) = A(f) = 0$, the two linear operators $\hat{L}$ and $\hat{B}$ are

$$\begin{aligned} \hat{L} &= \begin{pmatrix} 1+s & 0 \\ 0 & 1-s \end{pmatrix} \frac{\partial}{\partial x'} + \begin{pmatrix} 0 & f^* \\ f & 0 \end{pmatrix}, \\ \hat{B} &= -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x'^2} + \begin{pmatrix} |f|^2/(1+s) & if_{x'} \\ -if_{x'} & -|f|^2/(1-s) \end{pmatrix} \end{aligned} \tag{122}$$

where $s^2 = 1 - 2/b$, $x' = x\sqrt{2m/\eta^2}$. Thus the eigenvalue of NLSE is determined by

$$\hat{L}y = l\,y\ ,\quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{123}$$

Its corresponding solution can be found by use of inverse-scattering or another method.

The eigenequation corresponding to the NLSE (5) with V(x,t)= A(f)=0 and the Galilei invariance is found by the linear Zakharov-Shabat equation [31]:

$$iy_{x'} + \Phi y = l s_3 y \tag{124}$$

This equation is obtained from Eqs.(122) and (123), and is an eigenequation of eigenfunction y with an eigenvalue l and potential Φ, where,

$$s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Phi = \begin{pmatrix} 0 & f \\ f^* & 0 \end{pmatrix} \tag{125}$$

where f satisfies Eq.(5) with V(x,t)= A(f)=0. It evolves with time according to Eq. (121). However, what are the properties of the eigenvalue problems determined by these relations? This deserves further consideration.

As is known, the eigenequation is invariant under the Galilei transformation. As a matter of fact, if we substitute the following Galilei transformation:

$$f'(x',t') = e^{ivx' - iv^2/2} f(x',t'), x = x' - vt', t = t' \tag{126}$$

into Eq. (125), then Φ is transformed into

$$\Phi'(x') = \begin{pmatrix} e^{iq/2} & 0 \\ 0 & e^{-iq/2} \end{pmatrix} \Phi(x') \begin{pmatrix} e^{-iq/2} & 0 \\ 0 & e^{iq/2} \end{pmatrix} \tag{127}$$

where $q = vx' - \frac{1}{2}v^2 t' + q_0$, here q_0 is an arbitrary constant. If the eigenfunction $y(x')$ is transformed as

$$y'(x') = \begin{pmatrix} e^{iq/2} & 0 \\ 0 & e^{-iq/2} \end{pmatrix} y(x') \tag{128}$$

then Eq. (124) becomes

$$iy'_{x'} + \Phi' y' = (l - \frac{v}{2}) s_3 y' \tag{129}$$

It is clear that in the reference frame that is moving with the velocity v, the eigenvalue is reduced to v/2 compared with that in the rest frame. It shows that the velocity of the MIP (soliton) is given by $2\Re(l_k)$. When q is constant, i.e., $q = q_0$, the eigenvalue is unchanged

because v=0. This implies that the NLSE is invariant under the gauge transformation, $f' = e^{iq_0} f(x')$.

Satsuma and Yajima [70] studied the eigenfunction of Eq. (124) and its properties, where the eigenfunction satisfied the boundary condition, $y = 0$ at |x|→∞. The eigenvalues and the corresponding eigenfunctions were denoted by $l_1, l_2, ..., l_N and y_1, y_2, ..., y_N$, respectively. For a given eigenfunction , $y_n(x')$, equation (124) reads

$$i\frac{dy_n(x')}{dx'} + \Phi(x')y_n(x') = l_n s_3 y_n(x'), n = 1,2,...,N \tag{130}$$

$y(x')$ was expressed in terms of Pauli's spin matrices s_1 and s_2,

$$y(x') = \Re[y(x')]s_1 - \Im[y(x')]s_2 \tag{131}$$

Multiplying Eq. (130) by s_2 from the left and taking the transpose of the resulting equation, Satsuma et al [70]get

$$-i\frac{dy_m^T}{dx'}s_2 - y_m^T \Phi^* s_2 = i l_m y_m^T s_1 \tag{132}$$

where the superscript T denotes transpose. Multiplying the above equation by y_n from right and Eq. (129) by $y_m^T s_2$ from the left and subtracting one from the other, Satsuma and Yajima [70] obtained the following equation

$$(l_n - l_m)\int_{-\infty}^{\infty} y_m^T s_1 y_n dx' = 0$$

The boundary conditions, $y_n, y_m \to 0$ as $|x'| \to \infty$, were used in obtaining the above equation. Thus,the following orthonormal condition was then derived

$$\int_{-\infty}^{\infty} y_m^T s_1 y_n dx' = d_{nm} \tag{133}$$

Satsuma and Yajima further demonstrated that Eq. (130) has the following symmetry properties.

(i). If $f(x')$ satisfies $f(-x') = f^*(x')$, then replacing x' by - x' in Eq. (130) and multiplying again it by s_2 from left , we can get

$$i\frac{d}{dx'}[s_2 y_n(-x')]+\Phi(x')[s_2 y_n(-x')]=l_n s_3[s_2 y_n(-x')]$$

Since $s_2 y_n(-x')$ is also an eigenfunction associated with l_n, its behavior resembles that of $y_n(x')$ in the asymptotic region, i.e., $s_2 y_n(-x')\to 0$ as $|x'|\to\infty$, thus y_n has the following symmetry

$s_2 y_n(-x')=d\ y_n(x')$,or $y_n(-x')=d\ s_2 y_n(-x')$,($d=\pm 1$)

Therefore, if $f(-x')=f^*(x')$, then $y(x')$ satisfies the symmetry property $y_n(-x')=d\ s_1 y_n(-x')$ with $d=\pm 1$. This can easily be verified by replacing s_1 with s_2 in the above derivations.

(ii). If $f(x')$ is a symmetric (or antisymmetric) function of x', i.e., $f(-x')=\pm f(x')$, then $y_n^{(s)}(x')=s_1 y^*(-x')$ is the eigenfunction belonging to the eigenvalue $-l_n^*$, and $y_n^{(a)}(x')=s_2 y^*(-x')$ is the eigenfunction belonging to the eigenvalue l_n^*. The suffix s (or a) to the eigenfunction y_n indicates that f is symmetric (or antisymmetric). Since $f(-x')=f(x')$, replacing x' with - x' in Eq. (130) and taking complex conjugate, we get

$$i\frac{d}{dx'}[s_1 y^*(-x')]+\Phi(x')[s_1 y^*(-x')]=-l_n s_3[s_1 y^*_n(-x')]$$

Compared with Eq. (130), the above equation implies that $-l_n^*$ is also an eigenvalue and the associated eigenfunction $y_n^s(x')$ is just $s_1 y^*_n(-x')$, with the arbitrary constant. For $f(-x')=-f(x')$, the same conclusion is obtained by replacing s_1 with s_2 in the above derivations.

These symmetry properties are useful in providing a general view of the solution of Eq. (5) with V(x,t)= A(f)=0 . As is known, the real part of the eigenvalue, x_n, corresponds to the velocity of a soliton and the imaginary part, h_n, the amplitude. Then, if $f(x',t'=0)$, whose initial value has the symmetry $f(x',t'=0)$, =$\pm f(-x',t'=0)$, breaks into the series of solutions, the decay is bisymmetric, corresponding to the eigenvalues $-l_n^*$. If $f(x')$ is real, the above symmetry property yields

$$y_n^{(s)}(-x')=s_1[-ds_2 y_n^*(-x')]=ds_2 y_n^{(s)}(x')$$
$$y_n^{(a)}(-x')=s_2[-ds_1 y_n^*(-x')]=ds_1 y_n^{(a)}(x')$$

i.e., $y_n^{(s)}(x')$ has the same parity as $y_n(x')$, while $y_n^{(a)}(x')$ has the opposite one. When $f(-x')=-f(x')$ and l_n is pure imaginary ($l_n=-l_n^*$), the eigenvalues corresponding to the positive and negative parity eigenfunctions degenerate.

(iii). If $f(x')$ is real, but not antisymmetric, then the eigenvalue l_n is pure imaginary, i.e., $\Re(l_n)=0$. From Eq. (130) and its Hermitian conjugate, Satsuma et.al[68] found that

$$\Re(l_n)\langle n|s_2|n\rangle=\langle n|\Im[f(x')]s_3|n\rangle \tag{134}$$

with

$$\langle m|s_2|n\rangle=\int_{-\infty}^{\infty}y_m^+ s_2 y_n dx' \tag{135}$$

where $[\Phi,s_1]=2i\Im(f)s_3$ was used. We see from Eq. (134) that $\Re(l_n)$ vanishes if f is real and $\langle m|s_2|n\rangle\neq 0$. When f is a real and an antisymmetric function of x', the symmetry property (I) gives

$$\langle n|s_2|n\rangle=d^2\int_{-\infty}^{\infty}y_n^+(-x')s_1s_2s_1y_n(-x')dx'=-\langle n|s_2|n\rangle$$

Thus $\langle n|s_2|n\rangle=0$.

(iii). If the initial value takes the form of $f=e^{ivx'}R(x')$, where $R(x')$ is a real, but not antisymmetric function of x', then all the eigenvalues have the common real part, -v/2. This can be easily shown by the Galilei transformation. In fact, when $f(x',t'=0)=e^{ivx'}R(x')$, the solution does not decay to the series of solitons moving with the different velocities, but form a bound state. In this case, the real parts are common to all the eigenvalues, i.e., the relative velocities of the solitons vanish.

(iv). If f is a real non-antisymmertric function of x', it can be shown that

$$y_n^*(x')=ids_3y_n(x') \tag{136}$$

where $d=\pm 1$. Because $\Re(l_n)=0$, from the complex conjugate of Eq. (130), one can get $y_n^*(x')\propto s_3y$. Substituting Eq. (136) into the normalization condition Eq. (133), one then has $d=\pm 1$. If the eigenvalue of Eq. (124) is real, i.e., $l=x$ is real, then

$$i\frac{dy}{dx'}+\Phi y=xs_3y \tag{137}$$

and the adjoint function of y ,$\bar{y} = is_2 y^*$, is also a solution of Eq. (137), i.e.,

$$i\frac{d\bar{y}}{dx'} + \Phi\bar{y} = xs_3\bar{y}$$

From this and Eq. (137), Satsuma and Yajima obtained the following

$$\frac{d}{dx'}(y^+ y) = \frac{d}{dx'}(\bar{y}^+ y) = \frac{d}{dx'}(y^+ \bar{y}) = \frac{d}{dx'}(\bar{y}^+ \bar{y}) = 0 \qquad (138)$$

Using the above boundary conditions, they found that the solutions of Eq.(124) $y_1(x',x), y_2(x',x),$ and $\bar{y}_2(x',x)$ satisfy the following relations.

$$y_1^+ y_1 = y_2^+ y_2 = \bar{y}_2^+ \bar{y}_2 = 1, \bar{y}_2^+ y_2 = y_2^+ \bar{y}_2 = 0$$

From $y_1 = a(x)\bar{y}_2 + b(x)y_2$, we get a= $\bar{y}_2^+ y_1$ and b= $y_2^+ y_1$, where $y_1(x',x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ixx'}$, as $x' = -\infty$ and $y_2(x',x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{+ixx'}$, $\bar{y}_2(x',x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ixx'}$ as $x' = \infty$. As pointed out earlier, if a real (not antisymmetric) initial value is considered, the microscopic particle does not decay into moving solitons, but forms a bound states of solitons pulsating with the proper frequency. Satauma and Yajima developed a perturbation approach to investigate the conditions for the solutions to evolve and decay into moving solitons.

If the wave function f in Eq. (124) undergoes a small change, i.e., $f \to f' = f + \mathrm{V}f$, the corresponding change in Φ is given by

$$\Delta\Phi = \begin{pmatrix} 0 & \Delta f \\ \Delta f^* & 0 \end{pmatrix}.$$

Then, l_n and y_n changes as $l_n + \Delta l_n$ and $y_n + \mathrm{V}y_n$, respectively. To the first order in the variation, Eq. (124) becomes

$$[i\frac{d}{dx'} + (\Phi - l_n s_3)]\Delta y_n + (\Delta\Phi - \Delta l_n s_3)y_n = 0$$

Multiplying the above equation by $y_n^T s_2$ from the left and integrating with respect to x' over $(-\infty,\infty)$,Satauma et al[70] get

$$\Delta l_n = -i\int_{-\infty}^{\infty} y_n^T s_2 \Delta\Phi y_n dx' = -\int_{-\infty}^{\infty} y_n^T \Re(\Delta f) s_3 y_n dx' + i\int_{-\infty}^{\infty} y_n^T \Im(\Delta f) y_n dx'$$

If f is a real and non-antisymmetric function of x', Eq.(137) holds and

$$\Delta l_n = d < n|\Im(\Delta f)s_3|n> + id < n|\Re(\Delta f)|n> \tag{139}$$

Equation (139) indicates that if $< n|\Im(\Delta f)s_3|n> \neq 0$, the perturbation Δf makes the real part of the eigenvalue finite. That is, for the initial value, $f(x') + \Delta f(x')$, the solution of the NLSE in Eq.(5) with V(x,t)=A(f)=0 breaks up into moving solitons with velocity $2\Re(\Delta l_n)$. If f is real and is either a symmetric or an antisymmetric function of x', the above symmetry properties of eigenvalues of the NLSE lead to

$$< n|\Im(\Delta f(x'))s_3|n> = - < n|\Im(\Delta f(-x))s_3|n>$$

Therefore, if $\Im(\Delta f)$ is a symmetric function, $< n|\Im(\Delta f)s_3|n>$ vanishes, i.e., $\Re(\Delta l_n) = 0$, and the soliton bound state does not resolve into moving solitons even in the presence of the perturbation Δf.

Satsuma and Yajima[70] also obtained the shifts of the eigenvalues of Eq.(124) under the double-humped initial values, $f(x', t'=0) = f_0(x' - x_0') + e^{iq_0} f_0(x' + x_0')$, where f_0 is a real and symmetric function of x', x_0' and f_0 are real. The shifts of the eigenvalues were finally written as

$$\Delta l_n^{\pm} = d[\sin\ q < n | s_3 f_0(x' + 2x_0') | n > \mathrm{m}\sin(\frac{q_0}{2}) < n | s_3 f_0(x') e^{2x_0'(d/dx')} | n >] +$$

$$id[\cos q_0 < n | f_0(x' + 2x_0') | n > \pm \cos(\frac{q_0}{2}) < n | f_0(x') e^{2x_0'(d/dx')}) | n >]$$

where

$$-d\cos(\frac{q_0}{2}) < n | f_0(x') e^{2x_0'(d/dx')}) | n > + id\sin(\frac{q_0}{2}) < n | s_3 f_0(x') e^{2x_0'(d/dx')} | n >$$

$$= \int_{-\infty}^{\infty} y_2^{"(n)T} s_2 \Phi y_1^{"(n)} dx' = \int_{-\infty}^{\infty} y_1^{"(n)T} s_2 \Phi y_2^{"(n)} dx'$$

$$-d\cos(q_0) < n | f_0(x' + 2x_0^{"}) | n > -id\sin(q) < n | s_3 f_0(x' + 2x_0^{'}) | n >$$

$$= \int_{-\infty}^{\infty} y_1^{"(n)T} s_2 \Phi_2 y_1^{"(n)} dx' = \int_{-\infty}^{\infty} y_2^{"(n)T} s_2 \Phi_1 y_2^{"(n)} dx',$$

here

$$\Phi(x') = \Phi_1(x') + \Phi_2(x'), \Phi_1(x') = s_1 f_0(x' - x_0')$$

$$\Phi_2(x') = [(\cos(q_0) s_1 - \sin(q) s_2] f_0(x' + x_0')$$

The corresponding eigenvalue equation is given by

$$i\frac{d}{dx'} y_n'' + \Phi(x') y_n''(x') = l_n s_3 y_n''(x')$$

The eigenfunction $y_n^{"}(x')$ satisfies the following symmetry and orthogonality requirements:

$$y_{n\pm}''(-x') = \pm d[\cos(\frac{q_0}{2}) s_2 + \sin(\frac{q_0}{2}) s_1] y_{n\pm}''(x'), d \pm 1$$

$$\int_{-\infty}^{\infty} y_{n+}^{"T}(x') s_1 y_{n-}''(x') dx' = 0$$

When q_0 =0, $f(x')$ is real and symmetric, $\Delta l_n^{(\pm)}$ is pure imaginary, when $q_0 = p$, $f(x')$ is real and antisymmetric, $\Delta l_n^{(\pm)}$ is real,

$$\Re[\Delta l_n^{(\pm)}(q_0 = p)] = \mathrm{m}d < n | s_3 f_0(x') e^{2x_0'(d/dx')} | n >$$
$$\Im[\Delta l_n^{(\pm)}(q_0 = p)] = -d < n | s_3 f_0(x' + 2x_0') | n > \qquad (140)$$

Thus, the solution of the NLSE (5) with V(x,t)= A(f)=0 decays into paired solitons and each pair consists of solitons with equal amplitude and moving in the opposite directions with the same speed. For arbitrary $q_0^{'}$, we can see from Eq. (140) that the solution of Eq. (5) with V(x,t)= A(f)=0 breaks up into an even number of moving solitons with different speeds and amplitudes. Therefore, the eigenvalues of NLSE in nonlinear quantum mechanics are a very complicated problem.

5.Conclusions,the reasons establishing nonlinear quantum mechanics

In this paper we first presented the difficulties and problems of linear quantum mechanics. A main difficulty is that microscopic particle has only wave, not wave-particle duality, and disperses in total space.At the same time, we have to use $\left|y(\mathbf{r},t)\right|^2$ to represent the probability occurred particle at position $\vec{r}$, the mechanical quantities are denoted by average values, and the position and momentum of particle cannot be determinted simulatuously, and so on. These results are incompatible with the traditional concept of particles and de Broglie relations of wave-particle duality. Thus this result in long-time disputation in physics[7-9]. The roots generating these problems are that the quantum mechanics is too simplified and includes not important interactions among the particles or between the particles and background fields, thus, the Hamiltonian of the systems are not associated with the states of the particles. Therefore, it is very necessary to develop quantum mechanics for solving these problems. This is first reason why we want develop and establish NLQM.

When the above fuandamental hypothesises of LQM are broken through we established nonlinear quantum mechanics. Using its principles and theory, we study in detail some main properties of microscopic particles in nonlinear systems. We give the invariance and conservation laws of mass, energy and momentum and angular momentum for the microscopic particles, find also the classical laws of motion of microscopic particles and that motions of microscopic particles satisfy the Lagrangian and Hamilton equations. From dynamical equation –nonlinear Schrodinger equation and their solutions, the collision processes of many microscopic particles and their reflection and transmission features on the interface as well as the uncertainty relation of the motion of particles and quantum fluctuation of particle numbers, we obtained a lot of new properties of motion of MIPs, which are completely different from that in the linear quantum mechanics (LQM), for example, the particles possess the real wave-corpuscle duality, whose concrete image can be prefectly manifested by figure 1 and Eq.(9), obey the classical rule of motion and conservation laws of energy, momentum and mass, satisfy minimum uncertainty relation, can be localized due to the nonlinear interaction, and its position and momentum can be also determined in certain degree. Finally, we discuss further the eigenvalue problem of particles. These results are all compatible with the traditional concept of particles and de Broglie relations. This shows clearly that the nonlinear quantum mechanics established by us is correct. This is second reason why we want develop and establish NLQM.

Third reason establishing nonlinear quantum mechanics is that the nonlinear interaction, $b/2\,(ff^*)^2$, in Eqs.(5)-(6) is extensively existent in all physical systems including most simple hydrogen atom, only if we consider seriously the real motions of all particles and its interactions. For example, for motions of electrons, or excitons in the lattice in solid. we ever freezed the lattices and use a mean or periodic potential to describe the effects of the lattice field on the electrons or excitons in accordance with tranditional way in linear quantum mechanics. Very obviously, this neglects completely the practical motion of the lattice and simplified the interactions between them. If we consider exactly the effects and motions , then

the dynamical equations of the electron or exciton and the lattice should be, respectively, represented by

$$i\mathrm{h}\frac{\partial}{\partial t}f=-\frac{\mathrm{h}^2}{2m}\nabla^2 f+V(x,t)+cf\frac{\partial F}{\partial x} \qquad (141)$$

and

$$M(\frac{\partial^2 F}{\partial t^2}-v_0^2\frac{\partial^2 F}{\partial x^2})=-c\frac{\partial}{\partial x}|f|^2 \qquad (142)$$

where f denotes the state of a microscopic particle, such as electron, or exciton, Eq.(141) is its equation of motion. Equation(142) is the dynamics vibration of a background lattice field with velocity v_0, F denotes its displacement, c is a coupling interaction coefficient between the electron and background lattice field. The physical foudation of Eqs.(141)-(142) is as follows. When the displacement of the lattice is occurred, the state of the electron is changed through electron-phonon coupling interaction, then the electron or exciton moves in Eq.(141). However, the couteraction of the coupling changes also the state of vibration of the lattice, thus it moves in Eq.(142). From Eq.(142) we can find out

$$\frac{\partial F}{\partial x}=-\frac{c}{v^2-v_0^2}|f|^2 \qquad (143)$$

Inserting Eq.(143) into Eq.(141) yields the nonlinear Schrodinger equation (5) at $A(f)=0$, where $b=\dfrac{c^2}{M(v^2-v_0^2)}$. This is just a mechanism generating the nonlinear interaction.

In other nonlinear quantum systems the nonlinear interaction can generate by means of the following mechanism, i.e., the equation of motion of studied particle can be denoted by

$$i\mathrm{h}\frac{\partial}{\partial t}f=-\frac{\mathrm{h}^2}{2m}\nabla^2 f+V(x,t)f+cfF \qquad (144)$$

The equation of motion of other particles or field is represented by

$$M(\frac{\partial^2 F}{\partial t^2}-v_0^2\frac{\partial^2 F}{\partial x^2})=-c\frac{\partial^2}{\partial x^2}|f|^2 \qquad (145)$$

where F is the wave vector of other particle, c is a coupling coefficient between them. The relation between the two motion modes is

$$F=-\frac{c}{M(v^2-v_0^2)}|f|^2 \quad . \qquad (146)$$

Substituting Eq.(146）into Eq.(144) yieds also the nonlinear Schrodinger equation (5) with $A(f)=0$, where $b=\frac{c^2}{M(v^2-v_0^2)}$. The physical foundation of Eqs.(144)-(145) is as follows. When the vibration of the particles or displacement of field occurs, the state and probability distribution of the studied particle change due to the electron-phonon, or dipole-dipole or Coulomb interaction in atoms between them. Thus the studied particle moves in Eq.(144). However, the couteraction of the niteraction on the field or particle occur due to the fluctuation of gradient of density of the studied particle arising from the difference of the interaction at different sizes of displacement of the field. Thus the field is now in a forced vibration depicted by Eq.(145). This is just the physical significance of Eq.(145). The nonlinear interactions generating in atoms including hydrogen atom can be explained by this mechanism.

The above mechanisms and results show clearly the nonlinear interaction comes from the interactions among the particles or between the particles and background field. Since all realistic physics systems are composed of many particles and many bodies, the system composed only of one particle does not exist in nature. In such a case, the nonlinear interactions necessarily exist in any realistic physics systems including the hydrogen atom, only if we consider seriously the real motions of all particles and its interactions. Although the nonlinear interactions have different intensity in different systems, it exists always. Thus we cannot use linear quantum mechanics to study the features of motion of microscopic particles, even though the nonlinear interactions are very weak as mentioned above. This means that we should use nonlinear quantum mechanics in any a realistic physics systems. The linea quantum mechanics is only an approximate and linear theory and cannot correctly describe the states and properties of the microscopic particle in the physics systems of two or many badies.This again exhibits clearly the important significances for developing NLQM. This is also another reason why we want develop and establish NLQM.

Therefore, to develop and establish NLQM could solve problems disputed by scientists in the LQM field for about a century[7-9], can promote the development of physics and enhance and raise the knowledge and recognition levels to the essences of microscopic matter. We can predict that nonlinear quantum mechanics has extensive applications in physics, chemistry, biology, polymers, and so on.

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