The wave–corpuscle duality of microscopic particles depicted by nonlinear Schrödinger equation

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ABSTRACT

We used a nonlinear Schrödinger equation to replace the linear Schrödinger equation and to study the states of microscopic particles due to plenty of difficulties of quantum mechanics. From this investigation we find that the properties of microscopic particles are considerably changed relative to those in quantum mechanics. An unusual change is that the microscopic particles have a wave–corpuscle duality. The wave feature is followed from the solution of nonlinear Schrödinger equation, which is composed of an envelope and carrier waves, which propagate with determined frequency and velocity. The corpuscle feature is verified by the following results, i.e., the solutions have a mass centre and determinant size, mass, momentum and energy, which satisfy conservation laws of mass, momentum and energy, their collision obeys the collision law of classical particles, etc. The roots generating these changes are due to nonlinear interactions among the particles or between the particles and background field in the equation. The nonlinear interactions provide a double-well potential to make the microscopic particle localized as a soliton. Thus the Hamiltonian operator of the system breaks through the fundamental hypothesis of independence of wave function of states in quantum mechanics. This investigation indicates that the microscopic particle should be described by the nonlinear Schrödinger equation, instead of the linear Schrödinger equation, and the quantum mechanics should develop towards the direction of nonlinear domain.

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1. Introduction, difficulties of quantum mechanics

It is well known that the quantum mechanics established by several great scientists, such as, Bohr, Born, Schrödinger, Heisenberg, and others, in the early 1900s [1–6] is a foundation of modern science and used extensively to study the properties and rules of motion of microscopic particles. In this theory the states of microscopic particles are always described by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{r},t)\psi \tag{1}$$

where $\hbar^2\nabla^2/2m$ is the kinetic energy operator, $V(\vec{r},t)$ is the externally applied potential operator, m the mass of particles, $\psi(\vec{r},t)$ is a wave function describing the states of particles, $\vec{r}$ is the coordinate or position of the particle and t the time. Eq. (1) is a wave equation. Only if the externally applied potential is known can we find the solutions of the equation [7–9]. However, for all externally applied potentials, the solutions of the equation are only a linear or dispersive wave, for example, at $V(\vec{r},t) = 0$, its solution is a plane wave as follows:

$$\psi(\vec{r},t) = A' \exp[i(\vec{k}\cdot\vec{r} - \omega t)] \tag{2}$$

where k is the wavevector of the wave, ω is its frequency and A' its amplitude. This solution denotes the state of a freely moving microscopic particle with an eigenenergy of

$$E = \frac{p^2}{2m} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) \quad (-\infty < p_x, p_y, p_y < \infty) \tag{3}$$

This is a continuous spectrum of energy. It states that the probability for the particle to appear at any point in space is the same. Thus the microscopic particle propagates and distributes freely in a wave form in the entire space. This means that it cannot be localized. Thus it has nothing about corpuscle feature.

If the free particle is artificially confined in a small finite space, such as a rectangular box of dimension a, b and c, then the solutions of Eq. (1) are standing waves:

$$\psi(x,y,z,t) = A \sin\left(\frac{n_1\pi x}{a}\right)\sin\left(\frac{n_2\pi y}{b}\right)\sin\left(\frac{n_3\pi z}{c}\right)e^{-iEt/\hbar} \tag{4}$$

This shows that it still cannot be localized, and is always dispersive, namely, it appears still with a determinant probability

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at each point in the box with a quantized eigenenergy

$$E = \frac{\pi^2 \hbar^2}{2m} \left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2} \right) \qquad (5)$$

The corresponding momentum of the particle is also quantized. This means that the wave feature of microscopic particle has not been changed in this condition.

If the potential field is continuously varied, for example, the microscopic particle is subject to a conservative time-independent field, $U(\vec{r}, t) = U(\vec{r}) \neq 0$, then the microscopic particle satisfies now the time-independent linear Schrödinger equation

$$-\frac{\hbar^2}{2m} \nabla^2 \psi' + V(\vec{r}) \psi' = E \psi' \qquad (6)$$

where

$$\psi = \psi'(\vec{r}) e^{-iEt/\hbar} \qquad (7)$$

When $V = \vec{F} \cdot \vec{r}$, where $\vec{F}$ is a constant field, such as a one-dimensional uniform electric field with $V(x) = -e\varepsilon x$, the solution of Eq. (6) is

$$\psi' = A\sqrt{\xi} H_{1/2}^{(1)} \left(\frac{2}{3} \xi^{3/2} \right) \qquad \left(\xi = \frac{x}{l} + \lambda \right)$$

where $H^{(1)}(x)$ is the first-kind Hankel function, A is a normalized constant, l is the characteristic length and λ is a dimensionless quantity. The solution is still a dispersed wave. When $\xi \to \infty$, it approaches $\psi'(\xi) = A' \xi^{-1/4} e^{-2\xi^{3/2}/3}$, which is a damped wave.

If $V(x) = Fx^2$, the eigenenergy and eigenwave function are $E_n = (n + \frac{1}{2})\hbar\omega$ and $\psi'(x) = N_n e^{-a^2 x^2/2} H_n(\alpha x)$ $(n = 0, 1, 2, \ldots)$, respectively. Here $H_n(\alpha x)$ is the Hermite polynomial. The solution obviously has a decaying feature.

The above properties of solutions of Eq. (1) show clearly that the linear Schrödinger equation has only a wave solution, which has only a wave or dispersive feature, not the corpuscle property. Thus we have to use $|\psi(\vec{r}, t)|^2$ to represent the probability of particle occurring at position $\vec{r}$ and time t. At the same time, we see also that the wave feature of the microscopic particle cannot be changed with variations of time and external potential V. However, these features are incompatible and contradictory with regard to the traditional concept of particles [7–9]. Thus a great difficulty and trouble of quantum mechanics, such as the uncertainty relationship between position and momentum. The mechanic quantities of particles have only some average values in an any state. These difficulties are intrinsic and inherent in quantum mechanics, which result in a duration controversy in physics [8–12]. More surprising is that the way and method of solving these problems have not been sought up to now. Therefore it is very necessary to clarify the essence of these problems and to find the roots that generate these problems [8–12].

What is the reason causing these problems? As is known, the Hamiltonian operator of the system corresponding Eq. (1) is of the form

$$\hat{H}(t) = \hbar^2 \nabla^2 / 2m + V(\vec{r}, t) \qquad (9)$$

Obviously, it consists only of kinetic and potential operator of particles; the latter is not related to the state or wave function of the particle and is determined only by an externally applied field. We can keep changing the form of the external potential field $V(\vec{r})$, but we soon find out that the dispersion and decaying nature of the microscopic particle persist no matter what form the potential field takes. This means that the external potential field $V(\vec{r})$ can change only the shape of the microscopic particle, i.e., its

amplitude and velocity, but not its fundamental property, such as, the dispersion effect mentioned above. Therefore, the nature and features of microscopic particle are determined only by the kinetic energy term, $(\hbar^2/2m)\nabla^2 = \hat{p}^2/2m$, with dispersive effect, which cannot always be balanced and suppressed by an external potential field $V(\vec{r})$ in Eq. (1). Thus the particle has only the dispersive or wave feature. Because microscopic particles are always in motion, the dispersion effect of the kinetic energy term always exists. Thus, microscopic particles have permanently a wave or dispersive feature, not the corpuscle feature. This is just the root that the microscopic particles have only a wave feature in quantum mechanics.

The above root awakens and motivates us to seek for an interaction that can obstruct and suppress the dispersive effect of kinetic energy and make the microscopic particles eventually localized. Nonlinear interaction among the particles could play this role because it can also distort and collapse the dispersive wave [9–11]. Eventually the latter becomes a soliton with corpuscle feature when the two interactions are balanced and cancel with each other. In such a case, the nature of the particle can be changed and its feature of corpuscle can be displayed [13–18]. In the light of this idea a nonlinear interaction is added into Eq. (1). Thus the dynamic equation of microscopic particles should be replaced by the following nonlinear Schrödinger equation:

$$i\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \phi \pm b|\phi|^2 \phi + V(\vec{r}, t)\phi \qquad (10)$$

where $\phi(\vec{r}, t)$ is a wave function representing the states of microscopic particles and b a nonlinear interaction coefficient. The nonlinear interaction, $b|\phi|^2\phi$, related to the wave function of the particles is used here. Thus we expect that the nonlinear interaction can balance and suppress the dispersion effect of the kinetic term in Eq. (1) to make the particles localized. In the following we investigate the changes of nature of particles by Eq. (10).

2. Wave–corpuscle duality of microscopic particles

In the one-dimensional case, when $V(x, t) = 0$, Eq. (10) becomes

$$i\phi_{t'} + \phi_{x'x'} + b|\phi|^2 \phi = 0 \qquad (11)$$

where $x' = x/\sqrt{\hbar^2/2m}$, $t' = t/\hbar$. We now assume the solution of Eq. (11) to be of the form

$$\phi(x', t') = \varphi(x', t') e^{i\theta(x', t')} \qquad (12)$$

Inserting Eq. (12) into Eq. (11), we can obtain

$$\varphi_{x'x'} - \varphi\theta_{t'} - \varphi\theta_{x'}^2 - b\varphi^2\varphi = 0 \quad (b > 0) \qquad (13)$$

$$\varphi\theta_{x'x'} + 2\varphi_{x'}\theta_{x'} + \varphi_{t'} = 0 \qquad (14)$$

If we let $\theta = \theta(x' - v_c t')$, $\varphi = \varphi(x' - v_e t')$, then Eqs. (13) and (14), respectively, become

$$\varphi_{x'x'} - v_c\varphi\theta_{t'} - \varphi\theta_{x'}^2 - b\varphi^3 = 0 \qquad (15)$$

$$\varphi\theta_{x'x'} + 2\varphi_{x'}\theta_{x'} - v_e\varphi_{t'} = 0 \qquad (16)$$

On fixing the time t' and further integrating Eq. (16) with respect to x' we can get

$$\varphi^2(2\theta_{x'} - v_e) = A(t') \qquad (17)$$

Now let integral constant $A(t') = 0$. Then we can get $\theta_{x'} = v_e/2$. Again on substituting this into Eq. (15) and further integrating this

equation we then get

$$\int_{\varphi_0}^{\varphi} \frac{\mathrm{d}\varphi}{\sqrt{Q(\varphi)}} = x' - v_e t' \tag{18}$$

where $Q(\varphi) = -b\varphi^4/2 + (v_e^2 - 2v_c v_e)\varphi^2 + c'$.

When $c' = 0$ and $v_e^2 - 2v_c v_e > 0$, $\varphi = \pm\varphi_0$, $\varphi_0 = [2(v_e^2 - 2v_c v_e)/b]^{1/2}$ are the roots of $Q(\varphi) = 0$ except for $\varphi = 0$. Thus from Eq. (18) we obtain the solution of Eqs. (13) and (14) to be

$$\varphi(x', t') = \varphi_0 \operatorname{sech}\left[\sqrt{\frac{b}{2}}\varphi_0(x' - v_e t')\right]$$

Then the solution of nonlinear Schrödinger equation in Eq. (11) eventually is of the form

$$\phi(x, t) = A_0 \operatorname{sech}\left\{\frac{A_0\sqrt{b}}{\sqrt{2}\hbar}\left[\sqrt{2m}(x - x_0) - v_e t\right]\right\} e^{iv_e[\sqrt{2m}(x-x_0)-v_c t]/2\hbar} \tag{19}$$

Here $A_0 = \sqrt{(v_e^2 - 2v_c v_e)/2b}$. The solution in Eq. (19) can be found also by the inverse scattering method [13,14,16]. Obviously, this solution is completely different from Eq. (2), and contains an envelope and carrier waves. The former is $\varphi(x, t) = A_0 \operatorname{sech}\{A_0[\sqrt{2m}(x - x_0) - v_e t]/\sqrt{2}\hbar\}$, a bell-type non-topological soliton with amplitude A_0. The latter is $\exp\{iv_e[\sqrt{2m}(x - x_0) - v_c t]/2\hbar\}$; v_e is the group velocity of the particle and v_c the phase speed of the carrier wave. They are shown in Fig. 1. Therefore, the microscopic particle described by nonlinear Schrödinger equation (11) is a soliton [13–18]. Its envelope $\varphi(x, t)$ is a slow-varying function and the mass centre of the particle, the position of the mass centre, is just at x_0. A_0 is its amplitude and $W = 2\pi\hbar/(\sqrt{mb}A_0)$ is its width. Thus, the size of the particles is $A_0 W = 2\pi\hbar/\sqrt{mb}$, a constant. This shows that the particle has exactly a mass centre and determinant size, and is localized at x_0. For a certain system, v_e, v_c and size of the particle are determined and do not change with time. According to the soliton theory [13–18], the bell-type soliton in Eq. (19) can move over macroscopic distances with a uniform velocity v_e in space–time, retaining the form, energy, momentum and other quasi-particle properties. Just so, the vector $\vec{r}$ or x has

definitively physical significance, and denotes exactly the positions of the microscopic particles at time t. Then, the wavefunction $\phi(\vec{r}, t)$ or $\varphi(x, t)$ can represent exactly the states of the particles at the position $\vec{r}$ or x and time t. These features are consistent with the concept of particles. Thus the feature of corpuscle of microscopic particles is displayed clearly and outright.

At the same time, we show also the collision property of two soliton solutions of Eq. (11) by numerical simulation technique. The propagation of the soliton is indicated in Fig. 1(c). This figure shows clearly that the two particles can go through each other and can retain their form after the collision process. This feature is the same as that of classical particles. Therefore, the microscopic particle depicted by the nonlinear Schrödinger equation (11) has an explicitly corpuscle feature.

However, the envelope of the solution in Eq. (19) is a solitary wave. It has a certain wavevector and frequency as shown in Fig. 1(b), can propagate in space–time with the velocity v_e and is accompanied with the carrier wave. The feature of propagation depends only on the concrete nature of the particle. Fig. 1(b) shows the width of the frequency spectrum of the envelope $\varphi(x, t)$. The frequency spectrum has a localized structure around the carrier frequency ω_0.

The above results indicate clearly that the microscopic particle has exactly a wave–particulate duality [10–15]. This corresponds to Davisson and Germer's experimental result of electron diffraction on double seam in 1927 [6,8].

We can verify also that the wave–corpuscle duality of microscopic particles is not changed with externally applied potentials. Thus we calculate the solution of Eq. (10) and its feature at $V(x') = \alpha x' + c$, where α and c are some constants. In this case, Eq. (13) is replaced by

$$\varphi_{x'x'} - \varphi\theta_{t'} - \varphi\theta_{x'}^2 - b\varphi^2\varphi = \alpha x' + c \tag{20}$$

Now let

$$\varphi(x', t') = \varphi(\xi), \quad \xi = x' - u(t'), \quad u(t') = -\alpha(t')^2 + vt' + d \tag{21}$$

where $u(t')$ describes the accelerated motion of $\varphi(x', t')$. The boundary condition at $\xi \to \infty$ requires $\varphi(\xi)$ to approach zero

a

b

Φ

X_o $X-vt$

$\Delta\omega_o$

ω_o

topological solitary wcore of NLSE
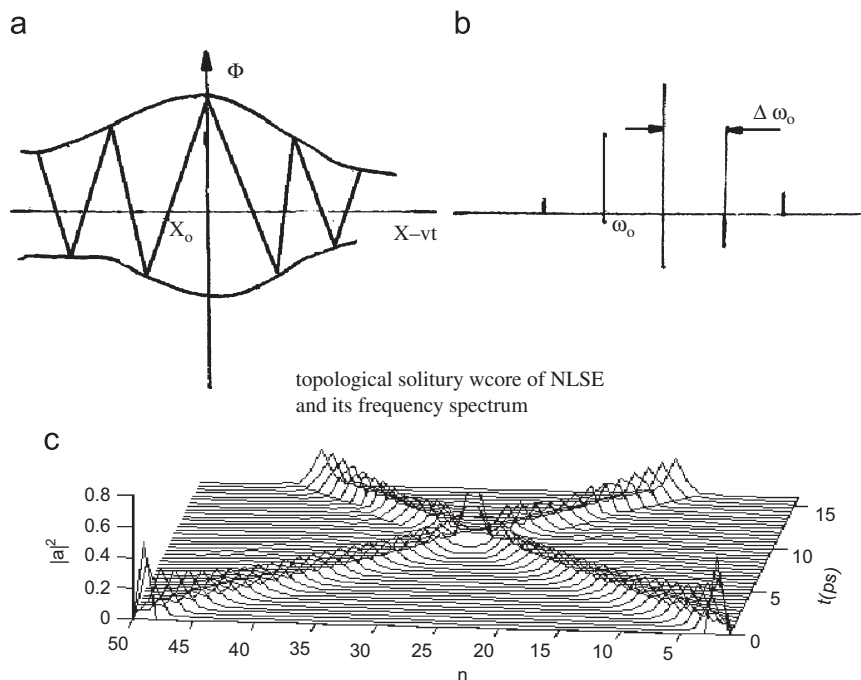and its frequency spectrum

c

Fig. 1. The solution Eq. (19) at $V = A = 0$ of Eq. (11) and its features.

rapidly. Eq. (14) in such a case may be written as

$$-\dot{u}\frac{\partial\varphi}{\partial\xi} + 2\frac{\partial\varphi}{\partial\xi}\frac{\partial\theta}{\partial\xi} + \varphi\frac{\partial^2\theta}{\partial\xi^2} = 0 \tag{22}$$

where $\dot{u} = \mathrm{d}u/\mathrm{d}t'$. If $2(\partial\theta/\partial\xi) - \dot{u} \neq 0$, Eq. (22) may be written as

$$\varphi^2 = \frac{g(t')}{(\partial\theta/\partial\xi - \dot{u}/2)} \quad \text{or} \quad \frac{\partial\theta}{\partial x'} = \frac{g(t')}{\varphi^2} + \frac{\dot{u}}{2} \tag{23}$$

Integration of Eq. (23) yields

$$\theta(x',t') = g(t')\int_0^{x'}\frac{\mathrm{d}x'}{\varphi^2} + \frac{\dot{u}}{2}x' + h(t') \tag{24}$$

where $h(t')$ is an undetermined constant of integration. From Eq. (24) we can get

$$\frac{\partial\theta}{\partial t'} = \dot{g}(t')\int_0^{x'}\frac{\mathrm{d}x'}{\varphi^2} - \frac{g\dot{u}}{\varphi^2} + \frac{g\dot{u}}{\varphi^2}\Big|_{x'=0} + \frac{\ddot{u}}{2}x' + \dot{h}(t') \tag{25}$$

Substituting Eqs. (24) and (25) in Eq. (20), we have

$$\frac{\partial^2\varphi}{\partial(x')^2} = \left[(\alpha x' + c) + \frac{\ddot{u}}{2}x' + \dot{h}(t') + \frac{\dot{u}^2}{4} + \dot{g}\int_0^{x'}\frac{\mathrm{d}x'}{\varphi^2} + \frac{g\dot{u}}{\varphi^2}\Big|_{x'=0}\right]$$
$$\times \varphi - b\varphi^3 + \frac{g^2}{\varphi^3} \tag{26}$$

Since $\partial^2\varphi/\partial(x')^2 = \mathrm{d}^2\varphi/\mathrm{d}\xi^2$, it is only a function of ξ. In order for the right-hand side of Eq. (26) to be also a function of ξ only, it is necessary that $g(t') = g_0 = \text{const}$:

$$(\alpha x' + c) + \frac{\ddot{u}}{2}x' + \dot{h}(t') + \frac{\dot{u}^2}{4} + \frac{g\dot{u}}{\varphi^2}\Big|_{x'=0} = \overline{V}(\xi) \tag{27}$$

Next, we assume that $V_0(\xi) = \overline{V}(\xi) - \beta$, where β is real and arbitrary. Then

$$\alpha x' + c = V_0(\xi) - \frac{\ddot{u}}{2}x' + \left[\beta - \frac{g\dot{u}}{\varphi^2}\Big|_{x'=0} - \dot{h}(t') - \frac{\dot{u}^2}{4}\right] \tag{28}$$

Clearly in the case being discussed, $V_0(\xi) = 0$, and the function in the brackets in Eq. (28) is a function of t'. Substituting Eqs. (27) and (28) in Eq. (26), we can get

$$\frac{\partial^2\tilde{\varphi}}{\partial\xi^2} = \beta\,\tilde{\varphi} - b\tilde{\varphi}^3 + \frac{g_0^2}{\tilde{\varphi}^3} \tag{29}$$

This shows that $\tilde{\varphi} = \varphi(\xi)$ is the solution of Eq. (29) when β and g are constants. For large $|\xi|$, we may assume that $|\tilde{\varphi}| \leqslant \beta/|\xi|^{1+\Delta}$, where Δ is a small constant. To ensure that $\mathrm{d}^2\tilde{\varphi}/\mathrm{d}\xi^2$ and $\tilde{\varphi}$ approach zero when $|\xi| \to \infty$, only the solution corresponding to $g_0 = 0$ in Eq. (29) is kept to be stable. Therefore we choose $g_0 = 0$ and obtain the following from Eq. (23):

$$\frac{\partial\theta}{\partial x'} = \frac{\dot{u}}{2} \tag{30}$$

Thus, we obtain from Eq. (28)

$$\alpha x' + c = -\frac{\ddot{u}}{2}x' + \beta - \dot{h}(t') - \frac{\dot{u}^2}{4},$$

$$h(t') = \left(\beta - c - \frac{1}{4}v^2\right)t' - \frac{1}{3}\alpha^2(t')^3 + v\alpha(t')^2/2 \tag{31}$$

Substituting Eq. (31) in Eqs. (24) and (25), we obtain

$$\theta = \left(-\alpha t' + \frac{1}{2}v\right)x' + \left(\beta - c - \frac{1}{4}v^2\right)t' - \frac{1}{3}\alpha^2(t')^3 + v\alpha(t')^2/2 \tag{32}$$

Finally, substituting the above in Eq. (29), we can get

$$\frac{\partial^2\tilde{\varphi}}{\partial\xi^2} - \beta\,\tilde{\varphi} + b\tilde{\varphi}^3 = 0 \tag{33}$$

When $\beta > 0$, the solution of Eq. (33) is of the form [13–17]

$$\tilde{\varphi} = \sqrt{\frac{2\beta}{b}}\mathrm{sech}(\sqrt{\beta}\xi) \tag{34}$$

Thus

$$\phi = \sqrt{\frac{2\beta}{b}}\mathrm{sech}\left[\sqrt{\beta}\left(\sqrt{\frac{2m}{\hbar^2}}(x-x_0) + \frac{\alpha t^2 - vt - d}{\hbar}\right)\right]$$
$$\times \exp\left\{\mathrm{i}\left[\left(\frac{-\alpha t}{\hbar} + \frac{v}{2}\right)\sqrt{\frac{2m}{\hbar^2}}x + \left(\beta - c - \frac{1}{4}v^2\right)\frac{t}{\hbar}\right.\right.$$
$$\left.\left. - \frac{\alpha^2 t^3}{3\hbar^3} + \frac{v\alpha t^2}{2\hbar}\right]\right\} \tag{35}$$

This is also a soliton solution. If $V(x') = C$, the solution can be represented as

$$\phi = \sqrt{\frac{2\beta}{b}}\mathrm{sech}\{\sqrt{\beta}[(x'-x'_0) - v_e(t'-t'_0)]\}$$
$$\times \exp\mathrm{i}\left\{\frac{v_e}{2\hbar}\left[(x'-x'_0) - \left(\beta - \frac{v_e^2}{4} - C\right)\right]t'\right\} \tag{36}$$

If $V(x') = ax'$ and $b = 2$, the solution can be represented as

$$\phi = 2\eta\,\mathrm{sech}[2\eta(x'-x'_0 - 4\xi t' + 2\alpha t'^2)]$$
$$\times \exp\left\{-\mathrm{i}\left[2(\xi - \alpha t')x' + \frac{4\alpha^2 t'^3}{3} - 4\alpha\xi t'^2 + 4(\xi^2 - \eta^2)t' + \theta_0\right]\right\} \tag{37}$$

In this case the transformation [13,14,19,20]

$$\phi(x',t') = \phi'(\tilde{x},\tilde{t})\mathrm{e}^{-\mathrm{i}\alpha\tilde{x}\tilde{t} - \mathrm{i}\alpha^2(\tilde{t})^3/3}, \quad x' = \tilde{x} - \alpha\tilde{t}^2, \quad t' = \tilde{t} \tag{38}$$

is used. Thus Eq. (10) becomes

$$\mathrm{i}\phi'_{\tilde{t}} + \phi'_{\tilde{x}\tilde{x}} + 2|\phi'|^2\phi' = 0$$

Utilizing Eq. (19), its solution in Eq. (37) can then be obtained immediately.

From Eqs. (35)–(38) we see clearly that the solutions of Eq. (10) for different potentials $V(x') = C$, $V(x') = ax'$ and $V(x') = ax' + C$ all consist of the envelope and carrier waves, the differences among them are only their amplitudes, velocities and phases. This means that the wave–corpuscle duality of the microscopic particles have not been changed, although the externally applied potentials are varied.

If a more complicated potential $V(x)$, for example $V(x) = kx^2 + A(t)x + B(t)$, is used in Eq. (10), the soliton solution of Eq. (10) can be also obtained in the light of the above method. It is

$$\phi = \varphi(x - u(t))\mathrm{e}^{\mathrm{i}\theta(x,t)} \tag{39}$$

where

$$\varphi(x - u(t)) = \sqrt{\frac{2B}{b}}\mathrm{sech}(a[(x-x_0) - u(t)]),$$

$$u(t') \equiv 2\cos(2\sqrt{k}t' + \beta) + u_0(t'),$$

$$\theta(x',t') = \left[-2\sqrt{k}\sin(2\sqrt{k}t' + \beta) + \frac{u_0}{2}\right] + \lambda_0 t' + g_0$$
$$- \int_0^1\left\{[u_0(t') - k(2\cos(2 - \sqrt{k}t' + \beta))]^2 + B(t')\right.$$
$$\left. + \left[\frac{u_0}{2} - 2\sqrt{k}\sin(2\sqrt{k}t' + \beta)\right]\right\}\mathrm{d}t'$$

When $A(t) = B(t) = 0$, $u(t') = 2\cos(2\sqrt{k}t') + u_0(t')$ and

$$\theta(x',t') = -2\sqrt{k}\sin\left(2\sqrt{k}t' + \frac{u_0}{2}x'\right) + g_0$$
$$- \int_0^1\left\{[-k(2\cos(2\sqrt{k}t') + u_0(t')]^2\right.$$
$$\left. + \left[\frac{u_0}{2} - 2\sqrt{k}\sin(2\sqrt{k}t')\right]\right\}\mathrm{d}t'$$

For the case of a harmonic potential, $V_0(x') = \alpha^2 x'^2$, where α is a constant. For this problem in accordance with above the solution can be written as [13,14,19,20]

$$\phi = 2\eta \, \text{sech}\left\{ 2\eta(x' - x_0') - \frac{4\xi\eta}{\alpha}\sin[2\alpha(t' - t_0')] \right\}$$
$$\times \exp\left\{ i\left[2\xi x' \cos 2\alpha(t' - t_0') - \frac{\xi^2}{\alpha}\sin[4\alpha(t' - t_0')] \right. \right.$$
$$\left. \left. +4\eta^2(t' - t_0') + \theta_0' \right] \right\} \tag{40}$$

where $2\sqrt{2/b}\eta = A_0$ and $2\sqrt{2}\xi = v_c$ are the amplitude and group velocity of the particles in Eqs. (38) and (40), respectively. We see that these solutions contain an envelope and carrier waves, which are the same as in Eqs. (19) and (35)–(38). Thus, the microscopic particle still has wave–corpuscle duality. At the same time, from these Eqs. (19), (35)–(38) and (40) we see that all these solutions of the nonlinear Schrödinger equation in Eq. (10) have the same shape as shown in Fig. 1 and similar nature, such as, containing an envelope, carrier waves and bell-type soliton with certain amplitude A_0, group velocity v_e and phase speed v_c. Meanwhile, these microscopic particles also have a mass centre and possess a certain amplitude, width and size. Thus they are all localized at x_0. Hence we can conclude that the wave–corpuscle duality of microscopic particles is independent of the externally applied potentials. The latter can change only the amplitude, velocity, frequency and phase. For instance, the velocity of particles is related to time in a certain case, but the frequency of the particle is oscillatory in another case. Therefore, the influence of potential in Eq. (10) is secondary. The fundamental nature of the particles is determined by the nonlinear interaction and kinetic energy terms in Eq. (10). The balance between them results in localization of the microscopic particle in the systems due to the fact that the nonlinear interaction suppresses the dispersive effect of the kinetic energy.

3. Invariance and conservation laws of mass, energy and momentum of particles

From the above results we see clearly that the microscopic particles are a soliton which can be denoted by $\phi(x', t') = \varphi(x', t')e^{i\theta(x', t')}$ in Eq. (12). According to the soliton theory [13–15], the bell-type soliton in Eq. (19) can move freely over macroscopic distances with a uniform velocity v_e in space–time, retaining its shape, energy, momentum and other quasi-particle properties. This means that its mass, momentum and energy are constants. The mass, momentum and energy in Eq. (19) can be represented by

$$N_s = \int_{-\infty}^{\infty} |\phi|^2 \, dx' = 2\sqrt{2}A_0,$$
$$p = -i\int_{-\infty}^{\infty} (\phi^*\phi_{x'} - \phi\phi_{x'}^*)dx' = 2\sqrt{2}A_0 v_e = N_s v_e = \text{const},$$
$$E = \int_{-\infty}^{\infty} \left[|\phi_{x'}|^2 - \frac{1}{2}|\phi|^4 \right] dx' = E_0 + \frac{1}{2}M_{sol}v_e^2 \tag{41}$$

where $x' = x/\sqrt{\hbar^2/2m}$ and $t' = t/\hbar$. $M_{sol} = N_s = 2\sqrt{2}A_0$ is the effective mass of the microscopic particle and a constant. This means that the energy, mass and momentum of the particle cannot be dispersed in its motion. Just so, the position vector $\vec{r}$ or position x in Eqs. (10) or (11) has definitive physical significance. Thus, the wave function $\phi(\vec{r}, t)$ or $\varphi(x, t)$ represents exactly the states of microscopic particles at the position $\vec{r}$ or x and time t. This is consistent with the concept of classical particles.

On the other hand, we know from classical physics that the invariance and conservation laws of mass, energy, momentum and angular momentum are some elementary and universal laws of matter, including classical particles. We can demonstrate that microscopic particles described by the nonlinear Schrödinger equation (10) also have such properties. Their mass, momentum and energy also satisfy the conservation laws. To verify this, we first give the Lagrangian $L = \int \mathscr{L} \, dx$ and Hamiltonian $H = \int \mathscr{H} \, dx$ of the particle systems corresponding to the nonlinear Schrödinger equation (10), where $\mathscr{L}$ is the Lagrangian density and $\mathscr{H}$ is the Hamiltonian density represented by wave function $\phi(x, t)$. Obviously, the wave function possesses derivatives of all orders and an integral, and is convergent and finite. The Lagrangian density function is of the form

$$\mathscr{L} = \frac{i\hbar}{2}(\phi^*\phi_t - \phi\phi_t^*) - \frac{\hbar^2}{2m}(\nabla\phi \cdot \nabla\phi^*) - V(x)\phi^*\phi + b/2(\phi^*\phi)^2 \tag{42}$$

The momentum density of this particle is defined as $p = \partial\mathscr{L}/\partial\phi$. Thus, the Hamiltonian density is

$$\mathscr{H} = \frac{i\hbar}{2}(\phi^*\partial_t\phi - \phi\,\partial_t\phi^*) - \mathscr{L} = \frac{\hbar^2}{2m}(\nabla\phi \cdot \nabla\phi^*) + V(x)\phi\phi^* - b/2(\phi\phi^*)^2 \tag{43}$$

From Eqs. (42) and (43), we see clearly that the Lagrangian and Hamiltonian functions of the systems corresponding to Eq. (10) involve a nonlinear interactional energy, $b/2(\phi\,\phi^*)^2$, related to the states of microscopic particles. This is in essence different from Eq. (9) in quantum mechanics. Then the nature and features of microscopic particles are simultaneously determined by the kinetic and nonlinear interaction terms in the nonlinear Schrödinger equation. Just so, there is a force or energy to obstruct and suppress the dispersive effect of kinetic energy in the system. Thus the particle cannot disperse and propagate freely in total space, and is localized all the time. This is just the essential reason that the microscopic particle has a particulate nature or corpuscle–wave duality as mentioned above. Therefore, we can say that the quantum systems described by nonlinear Schrödinger equation break through the fundamental hypothesis of the independence of Hamiltonian operator with wave function of particles in quantum mechanics. This is a new development in quantum theory.

In the general case, the total energy of the system is a function of t' and is represented by

$$E(t') = \int_{-\infty}^{\infty} \left[\left| \frac{\partial\phi}{\partial x'} \right|^2 - \frac{b}{2}|\phi\phi^*|^2 + V(x')|\phi|^2 \right] dx' \tag{44}$$

Thus, the number density, the number current and the densities of momentum and energy for the particle can be defined by

$$\rho = |\phi|^2, \quad p = -i\hbar(\phi^*\phi_x - \phi\phi_x^*),$$
$$J = i\hbar(\phi^*\phi_x - \phi\phi_x^*),$$
$$\varepsilon = \frac{\hbar^2}{2m}|\phi_x|^2 - \frac{b}{2}|\phi\phi^*|^2 + V(x)|\phi|^2 \tag{45}$$

where $\phi_x = (\partial/\partial x)\phi(x, t)$ and $\phi_t = (\partial/\partial t)\phi(x, t)$. From Eq. (10) and its conjugate equation as well as Eqs. (42)–(44) we can obtain

$$\frac{\partial p}{\partial t'} = \frac{\partial}{\partial x'}\left[2\left(\frac{\partial\phi}{\partial x'}\right)^2 + b|\phi\phi^*|^2 - 2V|\phi|^2 \right.$$
$$\left. -\left(\phi^*\frac{\partial^2\phi}{\partial x'^2} + \phi\frac{\partial^2\phi^*}{\partial x'^2}\right) + 2iV\left(\phi^*\frac{\partial\phi}{\partial x'}\right) \right],$$
$$\frac{\partial\rho}{\partial t'} = \frac{\partial J}{\partial x'}, \frac{\partial\varepsilon}{\partial x'} = \frac{\partial}{\partial x'}\left[\rho p + i\left(\frac{\partial\phi^*}{\partial x'}\frac{\partial^2\phi}{\partial x'^2} - \frac{\partial\phi}{\partial x'}\frac{\partial^2\phi^*}{\partial x'^2}\right) \right.$$
$$\left. -iV\left(\phi^*\frac{\partial\phi}{\partial x'} - \phi\frac{\partial\phi^*}{\partial x'}\right) \right]$$

Thus, we get the following forms for the integral of motion:

$$\frac{\partial M}{\partial t'} = \frac{\partial}{\partial t'} \int \rho \, dx' = 0, \quad \frac{\partial P}{\partial t'} = \frac{\partial}{\partial t'} \int p \, dx' = 0, \quad \frac{\partial E}{\partial t'} = \frac{\partial}{\partial t'} \int \varepsilon \, dx' = 0$$

(46)

These formulae represent just the conservation of mass, momentum and energy in such a case. This shows that the mass, momentum and energy of the microscopic particles described by the nonlinear Schrödinger equation Eq. (10) in the quantum systems still satisfy the conventional rules of conservation of matter in physics. Therefore, the microscopic particles described by the nonlinear Schrödinger equation Eq. (10) meet the common rules of motions of matter in nature. In the case of $V(x,t) = 0$ or constant, we can find out easily the values of mass, momentum and energy of the particles of Eq. (11) or (19) [13–18], which are shown in Eq. (41). These results show again that the microscopic particles in such a case have a corpuscle feature.

We can understand really from the above investigations the physical significance of wave function $\phi(\vec{r}, t)$ in Eq. (10). It represents in reality the states and properties of microscopic particles and $|\phi(x,t)|^2$ represents the number or mass density of particles. This is completely different from that of wave function in Eq. (1) in quantum mechanics, which denotes only a wave and $|\psi(\vec{r}, t)|^2$ expresses probability occurrence at a point in place–time. In the meanwhile, $\phi(\vec{r}, t)$ in Eq. (10) is always represented by Eq. (12), where $\varphi(x,t)$ and $\theta(x,t)$ are two independent physical quantities and represent the two different states of motion for envelope and carrier waves in the systems, respectively, but they are correlated with each other, which can see from Eqs. (13) and (14). The close correlation between them results in the soliton motion and wave–corpuscle duality of microscopic particles in the systems.

4. Roots of localization of microscopic particles and the effects of nonlinear interaction

From the above investigations we know that the nature of microscopic particles described by the nonlinear Schrödinger equation (10) is in essence different from those in quantum mechanics. In the former the microscopic particles have a wave–corpuscle duality. Evidently, the root occurring in this phenomenon is just the nonlinear interaction depending directly on the wave function of state of the particle. Thus it becomes possible to change the state and nature of the microscopic particle. Once the nonlinear interaction is so strong that it can balance and suppress the dispersion effect of the kinetic term in Eq. (10), the wave feature of the particle is suppressed and the shape of the wave becomes sech($x − vt$). The mass, energy and momentum of particles are concentrated. Thus the microscopic particle is localized at x_0 and becomes eventually a soliton with wave–corpuscle duality. No matter what the externally applied potential field $V(x)$ in Eq. (10) is, the nature of the microscopic particle cannot be changed.

However, what would be the effects of nonlinear interactions on microscopic particles? This is worth a deep study. To answer this, we first consider carefully motion of water wave in sea. When a wave approaches the beach, its shape varies gradually from a sinusoidal cross-section to triangular, and eventually to a crest, which moves faster than the rest. This is a result of the nonlinear nature of the wave. As the water wave approaches the beach the wave is broken up due to the fact that the nonlinear interaction is enhanced. In the nonlinear phenomenon, the speed of wave propagation depends on the height of the wave [14]. If the phase velocity of the wave, v_c, depends weakly on the height of the wave, h, then $v_c = \omega/k = v_{co} + \Theta_1 h$, where $\Theta_1 = \partial v_c/\partial h|_{h=h_0}$, h_0 is the average height of the wave surface, v_{co} is the linear part of the phase velocity of the wave and Θ_1 is a coefficient denoting the nonlinear effect. Therefore, the nonlinear interaction results in changes in both form and velocity of waves. This is the same for the dispersion effect, but their mechanism and rules are different. When the dispersive effect is weak, the velocity of a wave is denoted by $v_c = \omega/k = v'_{co} + \Theta_2 k^2$, where v'_{co} is a dispersionless phase velocity and $\Theta_2 = \partial^2 v_c/\partial k^2|_{k=k_0}$ is the coefficient of dispersion effect of the wave. Generally speaking, the lowest-order dispersion occurring in the phase velocity is proportional to k^2, and the term proportional to k gives rise to the dissipation effect. If the two effects act simultaneously on a wave, then the nature of the wave changes.

To further explore the effects of nonlinear interaction on the behaviors of microscopic particles, we consider a simple motion as follows:

$$\phi_e + \phi\phi_x = 0$$

(47)

where $\phi\phi_x$ is a nonlinear interaction. There is no dispersive term in this equation. It is easy to verify [14] that $\phi = \Phi'(x − \phi t)$ satisfies Eq. (47). This solution indicates that as time elapses, the front side of wave gets steeper and steeper, until it becomes triple-valued function of x due to the nonlinear interaction, which does not occur for a general wave equation. This is a deformation effect of wave resulting from the nonlinear interaction. If we let $\phi = \Phi' = \cos \pi x$ at $t = 0$, then at $x = 0.5$ and $t = \pi^{-1}$, $\phi = 0$ and $\phi_x = \infty$. The time $t_B = \pi^{-1}$ at which the wave becomes very steep is called the destruction period of the wave. However, the collapsing phenomenon can be suppressed by adding a dispersion term ϕ_{xxx} as in the KdV equation [13,14]. Then, the system has a stable soliton, sech$^2(X)$, in such a case. Therefore, a stable soliton, or localization of particle, can occur only if the nonlinear interaction and dispersive effect exist simultaneously in the system, and can be balanced and canceled against each other. Otherwise, the particle cannot be localized, and a stable soliton cannot be formed.

However, if ϕ_{xxx} is replaced by ϕ_{xx}, then Eq. (47) becomes

$$\phi_t + \phi\phi_x = \nu\phi_{xx} \quad (\nu > 0)$$

(48)

This is the Burger's equation. In such a case, the term $\nu\phi_{xx}$ cannot suppress the collapse of the wave arising from the nonlinear interaction $\phi\phi_x$. Therefore, the wave is damped. In fact, utilizing the Cole–Hopf transformation $\phi = -2\gamma(d/dx)(\log \psi')$, Eq. (48) becomes $\partial\psi'/\partial t = \nu(\partial^2\psi'/\partial x^2)$. This is a linear equation of heat conduction or diffusion equation, which has a damping solution. Therefore, the Burger's equation (48) is not an equation with soliton solution [13–17].

This example tells us that the deformational effect of nonlinearity on the wave can be suppressed only by the dispersive effect. Soliton solution of dynamic equations can then occur in such a case. The nonlinear term in nonlinear Schrödinger equation (10) sharpens the peak, while its dispersion term has the tendency to leave it off. Thus a soliton is formed. Localization of particle occurs in such a case. This example also verifies sufficiently that a stable soliton or localization of particle cannot occur in the absence of nonlinear interaction and dispersive effect or weak nonlinear interaction relative to the dispersive effect in the nonlinear Schrödinger equation in Eq. (10).

However, we also demonstrate that the solution of Eq. (10) is not the solution Eq. (2) of linear Schrödinger equation (1), even when the nonlinear interaction approaches zero. To see this clearly, we first examine the velocity of the skirt of the soliton given in Eq. (19). For weak nonlinear interaction ($b \ll 1$) and small skirt $\phi(x', t')$, it may be approximated by (for $x > v_e t$)

$$\phi = 4\sqrt{2/b}k e^{-2k(x' - v_e t')} e^{iv_e(x' - v_c t')/2}$$

(49)

where $2^{3/2}k/b^{1/2} = A_0$. Thanks to the small term $b|\phi|^2\phi$, Eq. (11) can be approximated by

$$i\phi_{t'} + \phi_{x'x'} \approx 0 \tag{50}$$

Substituting Eq. (49) into Eq. (50), we get $v_e \approx 4K$, which is the group speed of the particle. (Near the top of the peak, we must take both the nonlinear and dispersion terms into account because their contributions are of the same order. The result is the group speed.) Here, we have checked the formula only for the region where $\phi(x, t)$ is small; that is, when a particle is approximated by Eq. (49), it satisfies the approximate wave equation (50) with $v_e \approx 4K$.

However, if Eq. (50) is treated as a linear Schrödinger equation, its solution is of the form

$$\phi'(x', t') = A\,e^{i(kx' - \omega t')} \tag{51}$$

We now have $\omega = k^2$, which gives the phase velocity ω/k as $v_c = k$ and the group speed $\partial\omega/\partial k = v_{gr} = k$. Apparently, this is different from $v_e = 2\sqrt{2}k$. This is because the solution Eq. (49) is essentially different from Eq. (51). Therefore, the solution Eq. (51) is not the solution of nonlinear Schrödinger equation (10) with $V(x, t) = 0$ in the case of weak nonlinear interactions. Solution Eq. (49) is a "divergent solution" ($\phi(x, t) \to \infty$ at $x \to -\infty$), which is not an "ordinary plane wave". The concept of group speed does not apply to a divergent wave. Thus, we can say that the soliton is made from a divergent solution, which is abandoned in the linear waves. The divergence develops by the nonlinear term to yield solitary waves of finite amplitude. When the nonlinear term is very weak, the soliton diverges, and suppression of divergence results in no soliton. These circumstances are clearly seen from the soliton solution in Eq. (19) in the case of nonlinear coefficient $b \neq 1$. If the nonlinear term approaches zero ($b \to 0$), the solitary wave diverges ($\phi(x, t) \to \infty$). If we want to suppress the divergence, then we have to set $k = 0$. In such a case, we get Eq. (51) from Eq. (19). This illustrates that the nonlinear Schrödinger equation can reduce to the linear Schrödinger equation if and only if the nonlinear interaction and the group speed of the particle are zero. Therefore, we can conclude that the microscopic particles described by the nonlinear Schrödinger equation (10) in the weak-nonlinear-interaction limit is also not the same as that in linear Schrödinger equation in quantum mechanics. Only if the nonlinear interaction is zero, the nonlinear Schrödinger equation can reduce to the linear Schrödinger equation. However, real physical systems or materials are made up of a great number of microscopic particles. Nonlinear interactions arise from the interactions among the microscopic particles or between the microscopic particles and the environment, and exist always in the systems. Therefore, the nonlinear Schrödinger equation should be correct and more appropriate to describe the real systems, even in weak-non-linear-interaction cases. The linear Schrödinger equation (1) in quantum mechanics is an approximation to the nonlinear Schrödinger equation and can be used to study motions of microscopic particles in systems in which there exists only very weak and negligible nonlinear interactions.

However, how could a microscopic particle be localized in such a case? In order to shed light on the conditions for localization of microscopic particle in the nonlinear Schrödinger equation, we return to discuss the property of nonlinear Schrödinger equation (10). The time-independent solution of Eq. (10) is assumed to have the form [5–10]

$$\phi(x, t) = \varphi'(x, t)e^{-iEt/\hbar} \tag{52}$$

Then Eq. (10) becomes

$$-\frac{\hbar^2}{2m}\nabla^2\varphi' + [V(\vec{r}) - E]\varphi' - b|\varphi'|^2\varphi' = 0 \tag{53}$$

For the purpose of showing clearly the properties of this system, we here assume that $V(\vec{r})$ and b are independent of $\vec{r}$. Then in the one-dimensional case, Eq. (53) may be written as

$$\frac{\hbar^2}{2m}\frac{\partial^2\varphi'}{\partial x^2} = -\frac{d}{d\varphi'}V_{eff}(\varphi') \tag{54}$$

with

$$V_{eff}(\varphi') = \frac{1}{4}b|\varphi'|^4 - \frac{1}{2}(V - E)|\varphi'|^2 \tag{55}$$

When $V > E$ and $V < E$, the relationship between $V_{eff}(\varphi')$ and φ' is shown in Fig. 2. From this figure we see that there are two minimum values of the potential, corresponding to two ground states of the microscopic particle in the system, i.e., $\varphi'_0 = \pm\sqrt{(V - E)/b}$. This is a double-well potential, and the energy between the two ground states is $-(V - E)^2/4b \leqslant 0$. This shows that the microscopic particle can be localized due to the fact that the microscopic particle has negative binding energy. This localization is achieved through repeated reflection of the microscopic particle in the double-well potential field. The two ground states limit the energy diffusion. Thus the energy of the particle is concentrated, a soliton is formed and the particle is eventually localized. Obviously, this is a result of nonlinear interaction because the particle is in an expanded state if $b = 0$. In this case, there is only one ground state of the particle, which is $\varphi' = 0$. Therefore, the system can have two ground states only if $b \neq 0$, and the microscopic particle can be localized. Its binding energy, which makes the particle to be localized, is provided by the attractive nonlinear interaction, $-b/2(\varphi'\varphi'^*)^2$, in the systems.

From Eq. (55) we know that when $V > 0$, $E > 0$ and $V < E$ or $|V| > E$, $E > 0$ and $V < 0$ for $b > 0$. The microscopic particle may be not localized by virtue of the mechanisms mentioned above. On the other hand, we see from Eqs. (53)–(55) that if the nonlinear self-interaction is of repelling type (i.e., $b < 0$.), then Eq. (10) becomes

$$i\hbar\frac{\partial}{\partial t}\phi + \frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} - |b||\phi|^2\phi = V(x, t)\phi \tag{56}$$

It is impossible to obtain a bell-type soliton solution, with full matter features, from this equation. However, if $V(x, t) = V(x)$ or a constant, a solution of kink-type soliton exists. In this case, on inserting Eq. (52) into Eq. (56) we can get

$$\frac{\hbar^2}{2m}\frac{\partial^2\varphi'}{\partial x^2} - |b|\varphi'^3 + [E - V(x)]\varphi' = 0 \tag{57}$$

If V is independent of x and $0 < V < E$, then Eq. (56) has the following solution:

$$\varphi' = \frac{\sqrt{2(E - V)}}{|b|}\tanh\left[\sqrt{\frac{2(E - V)}{\hbar^2}}(x - x_0)\right] \tag{58}$$

This is the kink soliton solution when $|V| > E$ and $V < 0$. In the case of $V(x) = 0$, Zakharov and Shabat et al. [16] and Aossey et al. [21]
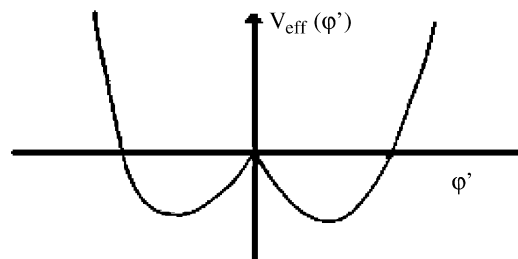

Fig. 2. The effective potential of nonlinear Schrödinger equation.

obtained the dark soliton solution, which was experimentally observed in optical fibers and was deeply discussed in the Bose–Einstein condensation model.

At present, a key question is how the negative nonlinear interaction in the nonlinear Schrödinger equation (10) is generated. In general, the nonlinear attraction can be produced by the following three mechanisms by means of interaction among particles or between the medium and particles. In the first mechanism, the attractive effect is due to interactions between the microscopic particles. This is called a self-interaction. A familiar example is the Bose–Einstein condensation mechanism of microscopic particles because of an attraction among the Bose particles. The mechanism is referred to as self-condensation. In the second mechanism, the medium has an anomalous dispersion effect (i.e., $k'' = \partial^2 k/\partial \omega^2|_{\omega_0 < 0}$) and nonlinear features resulting from anisotropy and nonuniformity. Thus the microscopic particles sense the nonlinear interaction through a modulation of nonlinear effects in the process of motion in the system. This mechanism is called self-focusing. The third mechanism is called self-trapping. It is produced by interaction between the microscopic particles and lattice or medium. However, irrespective of the mechanism and manner of production of nonlinear interactions, they always affect the interactions among the particles or between the particles and background field. In such a case their dynamical equations can be represented by

$$i\hbar \frac{\partial}{\partial t}\phi = -\frac{\hbar^2}{2m}\nabla^2\phi + V(x,t) + \chi\phi\frac{\partial F}{\partial x} \qquad (59)$$

and

$$\frac{\partial^2 F}{\partial t^2} - v_0^2\frac{\partial^2 F}{\partial x^2} = -\chi\frac{\partial}{\partial x}|\phi|^2 \qquad (60)$$

respectively, where ϕ denotes the state of a microscopic particle, F denotes the dynamics of a background field or another particle with velocity v_0 and χ is a coupling interaction coefficient between them. This coupling changes the states. From Eq. (60) we can find out that

$$\frac{\partial F}{\partial x} = -\frac{\chi}{v^2 - v_0^2}|\phi|^2 \qquad (61)$$

Inserting Eq. (61) into Eq. (59) yields

$$i\hbar \frac{\partial}{\partial t}\phi = -\frac{\hbar^2}{2m}\nabla^2\phi + V(x,t) - b|\phi|^2\phi$$

where $b = \chi^2/(v^2 - v_0^2)$. This equation is just the nonlinear Schrödinger equation of the microscopic particle in Eq. (10). This result shows clearly that the nonlinear interaction in the nonlinear Schrödinger equation (10) comes from the interactions among particles or between the particles and background field. Since all realistic physics systems are composed of many particles and many bodies, the system composed only of one particle does not exist in nature. In such a case, nonlinear interactions always exist in any realistic physics systems, including the hydrogen atom [10–15]. Therefore, when the states and properties of microscopic particles in a realistic physics system are studied we should use the nonlinear Schrödinger equation (10), instead of the linear Schrödinger equation (1) in quantum mechanics. Only if the coupling interaction is equal to zero or does not exist can Eq. (59) degenerate to the linear Schrödinger equation (1). This indicates again that the linear Schrödinger equation in quantum mechanics can describe the states and properties of only a single particle in vacuum without nonlinear interaction. However, such physical systems do not exist in nature. Therefore we conclude from this investigation that the linear Schrödinger equation is an approximate and linear theory and cannot correctly describe the states and properties of microscopic particle in a realistic system.

In previous investigations plenty of scientific workers used the linear Schrödinger equation (1) and quantum mechanics to study the states and properties of microscopic particles in systems of many particles and many bodies and obtained a lot of approximate results. However, since the linear Schrodiger equation (1) cannot describe the properties of microscopic particles in the system at all, we have to simplify these complicated and real nonlinear interactions among these particles by means of some approximate methods. In calculations, the nonlinear interactions determining the essence and nature of particles are always replaced by some simple and uniform average potentials unassociated with the states of particles. Thus the effects and results arising from these complicated and nonlinear interactions are ignored completely. Then the state and properties of particles obtained by the average potential are not real and correct states and properties of particles at all [22–25]. This shows that it is very necessary to re-study these problems by the nonlinear Schrödinger equation and corresponding quantum theory. This investigation indicates clearly that quantum mechanics very much needs to improve and develop towards the direction of nonlinear domain [24,25].

5. Conclusions

Since the states and properties of microscopic particles are being described by a linear Schrödinger equation, there are plenty of difficulties, which cause long-time and extensive controversies in quantum mechanics and have not been resolved up till now. In such a case we here used a nonlinear Schrödinger equation to replace it and to study further the nature and states of microscopic particles. From this investigation we find that the states and properties of microscopic particles are considerably changed relative to those in quantum mechanics. A key change is that the microscopic particles have a wave–corpuscle duality. This conclusion is obtained from the nature and properties of the solutions of nonlinear Schrödinger equation (10) with different external potentials, the significance of wave function and the conservation laws satisfied by mass, momentum and energy. The solution of the nonlinear Schrödinger equation (10) contains an envelope and carrier waves with determined frequency, which can propagate in the medium with a certain velocity. These display the wave feature of particle. However the solutions of Eq. (10) have a mass centre and possess a determined size, mass, momentum and energy, which also satisfy the conservation laws of mass, momentum and energy. Collisions of these particles obey the collision law of classical particles. These results embody the corpuscle feature of microscopic particles. Finally we seek the reasons and roots for generating these unusual properties and phenomena, which are due to nonlinear interactions among particles or between particles and background fields. Meanwhile, we verified that the linear Schrödinger equation can describe the states and properties of only a single microscopic particle in vacuum without nonlinear interaction. Quantum mechanics is an approximate and linear theory and cannot represent the properties and states of motion of microscopic particles in a realistic system properly. For a realistic system composed of many particles and many bodies we should use the nonlinear Schrödinger equation (10) to describe the states and properties of microscopic particles. The nonlinear interactions introduced in the nonlinear Schrödinger equation break through the fundamental hypothesis of independence of Hamiltonian operator of systems with wave function of states of particles in quantum mechanics [26,27]. Thus these microscopic particles are localized and have a real wave–corpuscle duality. Therefore our investigations point out the direction of development of quantum

mechanics and raise our knowledge and understanding of the essence and nature of microscopic particles.

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